ON p-ADIC LOOP GROUPS AND GRASSMANNIANS

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ABSTRACT. It is well-known that the coset spaces $G(k((z)))/G(k[[z]])$, for a reductive group G over a field k, carry a geometric structure, notably the structure of an ind-projective k-ind-scheme. This k-ind-scheme is known as the affine Grassmannian for G. From the point of view of number theory it would be interesting to gain an analogous geometric understanding of the quotients of the form $G(\mathrm{W}(k)[1/p])/G(\mathrm{W}(k))$, where W denotes the ring of Witt vectors. The present paper is an attempt to describe which constructions carry over from the 'function field case' to the 'p-adic case' and in particular to describe a construction of a p-adic affine Grassmannian for Sl_n as an fpqc-sheaf on the category of k-algebras for a perfect field k. In order to obtain a link with geometry we construct, inside a multigraded Hilbert scheme, projective k-schemes which map equivariantly to the p-adic affine Grassmannian inducing an isomorphism of Schubert cells and describe these morphisms on the level of k-valued points. Finally, we describe the R-valued points, where R is a perfect k-algebra, of the p-adic affine Grassmannian in terms of 'lattices over $\mathrm{W}(R)$', analogously to the function field case.

CONTENTS

1. INTRODUCTION

Let k denote an arbitrary field and let $G = \mathrm{Sl}_n$ denote the special linear group over k. The affine Grassmannian for G is constructed as an algebro-geometric model of the quotient $G(k((z)))/G(k[[z]])$. This means that one considers on the category of k-algebras the functor

$$\mathcal{G}rass : R \mapsto G(R((z)))/G(R[[z]])$$

(or rather the fpqc-sheaf associated with this functor) and obtains a description of $\mathcal{G}rass$ as an ind-scheme over k as follows. Recall that a lattice $L \subset R((z))^n$ is a finitely generated projective $R[[z]]$-submodule of $R((z))^n$ which satisfies $L \otimes_{R[[z]]}$

$R((z)) = R((z))^n$. Further, let N be any positive integer, and let $\mathcal{L}at_n^N(R)$ be the set of lattices L with the property that $z^N R[[z]]^n \subset L \subset z^{-N} R[[z]]^n$. Denote by $\mathcal{L}at_n^{N,0}(R) \subset \mathcal{L}at_n^N(R)$ the subset of special lattices – that is, lattices L with the additional property $\wedge^n L = R[[z]]$. In this situation Beauville and Laszlo ([BL94]) prove that $\mathcal{G}rass(R) = \cup_{N\in\mathbb{N}} \mathcal{L}at_n^{N,0}(R)$ and that the functor $\mathcal{L}at_n^{N,0}$ is represented by a closed subscheme of an ordinary Grassmannian (more precisely, the Grassmannian which parametrizes nN-dimensional k-linear subspaces in k^{2nN}). Hence the functor $\mathcal{G}rass$ is an ascending union of projective k-schemes, or, in other words, an ind-proper k-ind-scheme. This ind-scheme is the affine Grassmannian for $G = \mathrm{Sl}_n$. The affine Grassmannian, also for other algebraic groups than Sl_n, and its variants such as partial or full flag varieties, are well studied as natural objects within the geometric Langlands program, as well as for example in the theory of local models for certain Shimura varieties (see e.g. [Gör10]).

However, from the point of view of number theory it is perhaps even more natural to look at quotients of the form $G(\mathbb{Q}_p)/G(\mathbb{Z}_p)$, or more generally of the form $G(\mathrm{W}(k)[1/p])/G(\mathrm{W}(k))$, where k is a perfect field of positive characteristic and $\mathrm{W}(k)$ denotes the ring of Witt vectors over k. This would in particular lead to an algebraic model of the Bruhat-Tits building of the group $G(\mathrm{W}(k)[1/p])$. Let us refer to this setting as the 'p-adic case' in the following, while by the 'function field case' we mean the situation discussed in the preceding paragraph. Though there have been attempts to endow the quotient sets $G(\mathrm{W}(k)[1/p])/G(\mathrm{W}(k))$ with an algebro-geometric structure analogous to the one discussed above, e.g. by Haboush in [Hab05], the situation seems to be significantly more complicated in the p-adic case than in the function field case. One source of complication in the p-adic case is certainly the simple fact that $\mathrm{W}(R)$ (R any ring) does not carry a structure of R-module. This makes impossible the construction of an analogue of $\mathcal{L}at_n^{N,0}$ inside an ordinary Grassmannian, as described above for the function field case. Hence, the p-adic case is far worse understood, and it is still not clear whether it is possible to put a structure of ind-scheme, or a related algebraic structure, on the quotients $G(\mathrm{W}(k)[1/p])/G(\mathrm{W}(k))$. The present paper is an attempt to investigate how much of our understanding of the affine Grassmannian for Sl_n carries over from the function field case to the p-adic case.

In detail, we obtain the following results for the case $G = \mathrm{Sl}_n$: We define the 'p-adic loop group' $\mathrm{L}_p \mathrm{Sl}_n$ (an ind-scheme) and the 'positive p-adic loop group' $\mathrm{L}_p^+ \mathrm{Sl}_n$ (an affine scheme), and a natural morphism $\mathrm{L}_p^+ \mathrm{Sl}_n \to \mathrm{L}_p \mathrm{Sl}_n$. We construct the fpqc-quotient $\mathcal{G}rass_p := \mathrm{L}_p \mathrm{Sl}_n / \mathrm{L}_p^+ \mathrm{Sl}_n$, which we call the p-adic affine Grassmannian for Sl_n. Further, for every dominant cocharacter λ of the standard maximal torus $T \subset \mathrm{Sl}_n$, we identify a k-valued point of $\mathcal{G}rass_p$ (a certain 'lattice') and let $\mathcal{C}_\lambda$ be its $\mathrm{L}_p^+ \mathrm{Sl}_n$-orbit for the natural left-action of $\mathrm{L}_p^+ \mathrm{Sl}_n$ on $\mathcal{G}rass_p$. The $\mathcal{C}_\lambda$ play the role of Schubert cells in $\mathcal{G}rass_p$. Also, we construct for each λ a certain projective k-variety D_λ which carries a natural $\mathrm{L}_p^+ \mathrm{Sl}_n$-action, and such that D_λ contains an open orbit for this action. The varieties D_λ are related to the p-adic affine Grassmannian by the following

Theorem 1.1. *For every dominant cocharacter λ of the standard maximal torus $T \subset \mathrm{Sl}_n$ there is an $\mathrm{L}_p^+ \mathrm{Sl}_n$-equivariant morphism $\pi_\lambda : D_\lambda \to \mathcal{G}rass_p$ with the following properties: Let $C_\lambda \subset D_\lambda$ be the open orbit, and let $\mathcal{C}_\lambda \subset \mathcal{G}rass_p$ be the Schubert cell corresponding to λ. Then π_λ induces an isomorphism of functors*

$C_\lambda \simeq \mathcal{C}_\lambda$, and in particular the Schubert cells of $\mathcal{G}rass_p$ are quasi-projective k-schemes. Moreover, the image under π_λ of D_λ is precisely the union of the sets of k-valued points of the Schubert cells indexed by λ' with $\lambda' \leq \lambda$ for the Bruhat-order.

However, the morphisms π_λ are not injective on the level of k-valued points as one might hope (in order to obtain p-adic analogues of Schubert *varieties*). By a short example and by the relationship with constructions in [Kre10] we argue that the varieties D_λ should perhaps better be viewed as an analogue of Demazure resolutions in the p-adic setting.

Finally, we answer the following question: What is the set of R-valued points of the p-adic affine Grassmannian for a perfect k-algebra R? In close analogy to the function field case we give the definition: If R denotes a perfect k-algebra, then a lattice $L \subset \mathrm{W}(R)[1/p]^n$ is a finitely generated projective $\mathrm{W}(R)$-submodule with the property that $L \otimes_{\mathrm{W}(R)} \mathrm{W}(R)[1/p] = \mathrm{W}(R)[1/p]^n$. We prove the following

Theorem 1.2. *If R is a perfect k-algebra, then the set of R-valued points of $\mathcal{G}rass_p$ is equal to the set of lattices $L \subset \mathrm{W}(R)[1/p]^n$ with $\wedge^n L = \mathrm{W}(R)$.*

As a corollary we obtain a characterization of lattices in $\mathrm{W}(R)[1/p]^n$: A $\mathrm{W}(R)$-submodule $L \subset \mathrm{W}(R)[1/p]^n$ is a lattice if and only if it is free Zariski-locally over R if and only if it is free fpqc-locally over R. This is again analogous to a well known characterization of lattices in the function field case (see Beauville-Laszlo [BL94] or Görtz [Gör10]).

Here is a more detailled outline of the paper: After briefly reviewing the notion of ind-scheme in Section 2, we recall in Section 3 Greenberg's notion of 'realization', which he introduces in [Gre61]. Implicitly, Greenberg's notion of realization appears whenever one speaks about (formal) loop groups and their quotients, i.e. affine Grassmannians. I have tried in this paper to carry out constructions explicitly within this formal framework of Greenberg realizations, and thus we recall in this section Greenberg's definition of realization and basic results on their existence and functorial properties. Moreover, we give a definition of 'localized' Greenberg realization which will serve our purpose of constructing 'generalized' loop groups (e.g. p-adic ones) in terms of ind-schemes. These constructions are used in Section 4 to define and study the notions of (positive) loop space resp. (positive) loop group associated with schemes resp. groups over discrete valuation rings.

In Section 5, our version of a p-adic Grassmannian for Sl_n, $\mathcal{G}rass_p$, is introduced as the fpqc-quotient of the p-adic loop group $\mathrm{L}_p\, G$ by the positive p-adic loop group $\mathrm{L}_p^+\, G$. Schubert cells $\mathcal{C}_\lambda \subset \mathcal{G}rass_p$ are defined, as in the function field case, as $\mathrm{L}_p^+\, G$-orbits of lattices corresponding to dominant cocharacters λ of the standard maximal torus T of Sl_n. As expected, the set of k-valued points of the p-adic affine Grassmannian is the disjoint union of the sets $\mathcal{C}_\lambda(k)$, $\lambda \in \check{\mathrm{X}}_+(T)$.

In the following Section 6 we construct the projective k-varieties D_λ, where λ is a dominant cocharacter of T. The strategy follows the natural guess for the construction of the p-adic analogue of Schubert varieties: Via Greenberg realization we consider the underlying set of the free $\mathrm{W}_N(k)$-module $\mathrm{W}_N(k)^n$ as the set of k-valued points of an affine space $\mathbb{A}_k^{nN}$, and also we realize the module operations on $\mathrm{W}_N(k)^n$ as morphisms 'of $\mathbb{A}_k^{nN}$'. The choice of a certain natural grading on the coordinate ring of $\mathbb{A}_k^{nN}$ makes all these morphisms *graded*. By a result of Haiman and Sturmfels there exists a universal flat family of graded submodules, parametrized by a (projective) 'multigraded Hilbert scheme' H over k, and we

show that there exists a closed k-subscheme $Z \subset H$ which parametrizes those submodules which are stable under the module operations on $\mathbb{A}_k^{nN}$. The $\mathrm{L}_p^+ \mathrm{Sl}_n$-operation on $\mathbb{A}_k^{nN}$ induces an action on H which restricts to Z. Now D_λ is defined as the orbit closure of a certain k-valued point of Z which corresponds to the lattice $\mathrm{diag}(p^\lambda) \cdot \mathrm{W}(k)^n$. In other words, the role of the ordinary Grassmann varieties in the construction of Schubert varieties for the affine Grassmannian is now played by a multigraded Hilbert scheme. A very similar strategy is also pursued by Haboush in [Hab05].

The following Sections 7 and 8 are devoted to the proof of Theorems 1.1 and 1.2.

Finally, in the appendix we collect a couple of easy resp. standard results on fpqc-sheaves and fpqc-sheafification which are used throughout the paper. Moreover we discuss very briefly the set-theoretical problems which occur when talking about fpqc-sheafifications, and which are often ignored. Using results of Waterhouse, [Wat75], we check that such complications do not occur in our construction of the p-adic affine Grassmannian as an fpqc-sheaf quotient of loop groups.

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2. Ind-schemes

Throughout this paper, k denotes a field. In this paper we make extensive use of the language of ind-schemes. Since there are different definitions of the term 'ind-scheme' scattered through the literature let us begin by fixing terminology and giving a brief discussion of our notion of ind-scheme.

Definition 2.1. Let S be a scheme. An *S-space* is a sheaf on the fpqc-site over S. An *ind-scheme over S* (or *S-ind-scheme*, or simply *ind-scheme*) is the colimit in the category of S-spaces of a direct system of quasi-compact S-schemes. Morphisms of ind-schemes are morphisms of functors.

If an S-ind-scheme X has the form $X = \varinjlim_{i \in I} X_i$ with all the X_i quasi-compact, then we say that X is *represented* by the direct system $(X_i)_{i \in I}$. By abuse of language we will also simply speak of the ind-scheme (X_i).

If X is an ind-scheme, then by a *sub-ind-scheme* $Y \subset X$ we mean a subfunctor of X which is itself an ind-scheme. A sub-ind-scheme $Y \subset (X_i)_i$ is called *ind-closed*, if it is represented by a system of closed subschemes $Y_i \subset X_i$.

Throughout this paper we will assume the directed index set I to be countable. In particular, there always exists a cofinal subset $I' \subset I$ which can be identified with the natural numbers. We denote the category of S-spaces by (S-Sp) (the morphisms between two S-spaces being natural transformations of functors), and by (ind-Sch/S) we denote its full subcategory whose objects are the S-ind-schemes. In other words, we have the following fully faithful functors:

$$(\mathrm{Sch}/S) \hookrightarrow (\text{ind-Sch}/S) \hookrightarrow (S\text{-Sp})$$

Remark 2.2. (1) Our definitions of S-space and S-ind-scheme coincide with those given by Beauville and Laszlo in [BL94] in the case where S=Spec k for some field k. There are different definitions for these terms, e.g. by Drinfel'd in [Dri03].

(2) The existence of colimits in the category of S-spaces of a of direct system of S-schemes needs a little justification, which is given in the appendix of this paper (Proposition 9.5). In fact, sheafification of an arbitrary presheaf for the fpqc-topology poses set theoretical problems (Waterhouse, [Wat75]), which we also discuss briefly in the appendix. The need for fpqc-sheafification will arise again when we introduce our p-adic version of the affine Grassmannian.

Let us collect a few easy facts about ind-schemes.

Lemma 2.3. *If T is a quasi-compact scheme and X is an ind-scheme which is represented by a direct system (X_i), then $Mor(T,X) = \varinjlim Mor(T, X_i)$.* □

Proof. As we prove in the appendix (Proposition 9.5), the ind-scheme X is just the Zariski-sheafification of the presheaf-direct limit $\varinjlim X_i$. Since every Zariski-covering of a quasi-compact T has a finite subcovering, the lemma follows. □

Let X and Y be ind-schemes which are represented by direct systems (X_i) and (Y_i), respectively. Any morphism of direct systems $(X_i) \to (Y_i)$ (i.e. a system of compatible maps $f_i : X_i \to Y_{i'}$) induces a morphism $f : X \to Y$. In this case we say that f is represented by the system (f_i). From the above lemma the following converse is easy to deduce.

Lemma 2.4. *Let X and Y be ind-schemes which are represented by direct systems (X_i) and (Y_i), respectively. Then every morphism $X \to Y$ is represented by a compatible system of maps $f_i : X_i \to Y_{i'}$.* □

Note that this lemma holds precisely because quasi-compactness of all the X_i is built in the definition of ind-scheme resp. representing direct systems. Moreover, as remarked above, we can always assume that all our index sets are equal to the set of natural numbers, and that compatible systems of maps are of the form $f_i : X_i \to Y_i$ (i.e. preserve the index).

Lemma 2.5 (Products)**.** *Let X, Y, Z be ind-schemes which are represented by direct systems $(X_i), (Y_i), (Z_i)$, respectively, and let $X \to Z$ and $Y \to Z$ be morphisms represented by compatible systems of maps $X_i \to Z_i$ and $Y_i \to Z_i$. Then the fiber product (in the category of k-spaces) $X \times_Z Y$ is an ind-scheme and is represented by the direct system $(X_i \times_{Z_i} Y_i)$.* □

We will make a further assumption to simplify our presentation. *Throughout this paper, all test-schemes which occur will be assumed to be quasi-compact.* In other words, all functors are considered to be functors on categories of quasi-compact schemes. This simplification is justified by the fact that an S-space is determined by its values on quasi-compact (or even affine) S-schemes. Thus we will not further distinguish between the ind-scheme represented by a direct system (X_i) and the presheaf-direct limit $\varinjlim X_i$.

3. Greenberg realizations

In this section we recall Greenbergs notion of 'realization' (in the category of schemes) and introduce the notion of 'localized Greenberg realization' in the category of ind-schemes.

3.1. **Greenberg realizations.** Our reference for this is Greenberg [Gre61]. We will stay close to Greenberg's notation, and in particular *in this section we use the letter R to denote a ring scheme.* So let S be a scheme and $R \to S$ a ring scheme over S. Hence R represents a sheaf of rings on the Zariski-site over S, and thus defines a covariant functor

$$G_R : (\mathrm{Sch}/S) \to (\text{Ringed spaces}/\operatorname{Spec} R(S))$$
$$(X, \mathcal{O}_X) \mapsto G_R(X) = (X, \mathcal{O}_{G_R(X)}),$$

where $\mathcal{O}_{G_R(X)}(U) := R(U)$, the set of S-morphisms from U to R. The ring scheme R is called a *local* ring scheme, if the functor G_R takes values in the category of locally ringed spaces.

Example 3.1. Let $R = \mathrm{W}_N$ be the scheme of Witt-vectors of length N over $S = \operatorname{Spec} k$, with $0 \leq N \leq \infty$. We claim that W_N is a local ring scheme. Namely, for any S-scheme X the stalk of $G_{\mathrm{W}_N} X$ at $x \in X$ is given by $\mathcal{O}_{G_{\mathrm{W}_N}(X),x} = \varinjlim \mathrm{W}_N(U)$, and $f = (f_0, f_1, \dots) \in \mathcal{O}_{G_{\mathrm{W}_N}(X),x}$ is invertible if and only if $f_0 \in \mathcal{O}_{X,x}$ is invertible. The 'only if'-part is trivial, and the 'if'-part can be seen as follows. Whenever f_0 is invertible in $\mathcal{O}_{X,x}$, then there exists an open neighbourhood U of x such that f_0 is invertible in $\mathcal{O}_X(U)$. But then f is invertible in $\mathrm{W}_N(U)$ and a fortiori in $\mathcal{O}_{G_{\mathrm{W}_N}(X),x}$.

The situation of this example, R being the scheme of Witt vectors of finite or infinite length over a perfect field k, will be the most interesting for us, as we are aiming towards the construction of p-adic loop groups. Another familiar example of local ring scheme is the scheme of power series in one variable over k, i.e. the scheme $\mathbb{A}_k^\infty$ endowed with the structure of ring scheme so that $\mathbb{A}_k^\infty(A) = A[[z]]$ for any k-algebra A.

In the following let R be a local ring scheme over S.

Definition 3.2 (Greenberg, [Gre61])**.** Let X be a scheme over the ring $R(S)$. A *(Greenberg) realization of X over S* is an S-scheme $F_R X$ which represents the functor

$$Y \mapsto \mathrm{Hom}_{l.r.sp./R(S)}(G_R(Y), X).$$

In the sequel, to simplify notation, we will occasionally drop the index indicating the ring scheme R.

The following proposition and its corollary are purely formal consequences of the universality of representing objects. However, since these are especially interesting for our applications in the construction of loop groups, we state them explicitly:

Proposition 3.3. *Realizations commute with fiber products. More precisely, if X, X', T are $R(S)$-schemes having realizations FX, FX', FT over S, then $FX \times_{FT} FX'$ is a realization over S of $X \times_T X'$.* □

Corollary 3.4. *Let X be a group scheme over $R(S)$ having a realization FX over S. Then FX is a group scheme over S.* □

Let us now explicitly describe realizations in situations which are of interest for us (as always, we keep in mind the situation where $S = \operatorname{Spec} k$, and the local ring scheme R is the scheme of Witt vectors of finite or infinite length). Detailed proofs are presented in [Gre61].

Proposition 3.5 (Greenberg, [Gre61])**.** *Assume that there is an isomorphism of S-schemes*

$$\varphi = (\varphi_1, \dots, \varphi_N) : R \to \mathbb{A}^N_S,$$

where $0 \le N \le \infty$. Let

$$\varphi = (\varphi_1, \dots, \varphi_N) : R \to \mathbb{A}^N_S$$

be such an isomorphism. Then $\mathbb{A}^d_{R(S)}$ has as a Greenberg realization the S-scheme $F(\mathbb{A}^d_{R(S)}) = (\mathbb{A}^N_S)^d$ together with the universal arrow $\lambda : GF(\mathbb{A}^d_{R(S)}) \to (\mathbb{A}^d_{R(S)})$ given in terms of global sections by the ring homomorphism

$$\begin{aligned} \lambda^\# : R(S)[T_1, \dots, T_d] &\to (S[t_{1,1}, \dots, t_{1,N}, \dots, t_{d,1}, \dots, t_{d,N}])^N \\ T_i &\mapsto (t_{i,1}, \dots, t_{i,N}). \end{aligned}$$

If $f : \mathbb{A}^d_{R(S)} \to \mathbb{A}^e_{R(S)}$ is a morphism of $R(S)$-schemes and $P_1, \dots, P_e$ are the polynomials in $R(S)[T_1, \dots, T_d]$ defining f, then the morphism Ff between the respective Greenberg realizations is given in terms of global sections by

$$t'_{i,j} \mapsto \varphi_j(\lambda^\#(P_i)).$$

Here, the $t_{i,j}$ are the coordinates on $F(\mathbb{A}^d_{R(S)})$, while the $t'_{i,j}$ are the coordinates on $F(\mathbb{A}^e_{R(S)})$. In other words, to calculate the image of $t'_{i,j}$, we have to substitute $T_l \mapsto (t_{l,j})_j$ in the polynomial P_i and then take the j-th component of the result under the isomorphism φ.

Proof. This is proved by Greenberg in [Gre61] in the case where N is finite. However, his proof carries over to the situation $N = \infty$ without changes. □

Proposition 3.6 (Greenberg, [Gre61])**.** *Let R be a local ring scheme over S, being isomorphic as an S-scheme to an N-dimensional affine space over S (recall that we allow $N = \infty$). Let moreover X be an affine scheme of finite type over $R(S)$ having a realization by an affine scheme FX over S. Then every closed subscheme of X has a realization over S by a closed subscheme of FX.*

Proof. This is proved in [Gre61]. The crucial point of the proof is the observation that we may, by universality of realizations, assume that X itself is an affine space over $R(S)$, and that we obtain a realization of X as follows. Let $X = \mathbb{A}^d_{R(S)}$ and choose a set of defining equations $f_i(X_1, \dots, X_d)$ for a closed subscheme $Y \subset X$. Then each X_j can be understood as a vector of coordinates $X_i = (x_{i,0}, \dots, x_{i,N})$ (according to the isomorphism $R(S) \simeq \mathbb{A}^N_S(S)$). Plugging these into the equations $f_i = 0$ yields 'coordinate-wise' equations in the variables $x_{i,j}$. These are the defining equations of $FY \subset FX$. □

Let us consider for instance the case $R = \mathrm{W}_N$. Let X be the affine space $\mathbb{A}^d_{\mathrm{W}_N(S)} = \operatorname{Spec} \mathrm{W}_N(S)[T_1, \dots, T_d]$. Then a closed subscheme $X \subset \mathbb{A}^d_{\mathrm{W}_N(S)}$ is given by a set of equations, say

$$\{f_1(T_1, \dots, T_d), f_2(T_1, \dots, T_d) \dots\}.$$

The equations of the realization $FX \subset \operatorname{Spec} S[t_{i,j}]$ are then obtained by plugging the Witt vectors

$$(t_{i,0}, t_{i,1}, \dots) \in W_N(S[t_{i,0}, t_{i,1}, \dots])$$

into the equations f_m. The components of the Witt vectors

$$f_m(t_{i,0}, t_{i,1}, \dots) \in W_N(S[t_{i,0}, t_{i,1}])$$

for varying m are the defining equations of the realization FX.

3.2. **Localized Greenberg realizations.** Let R be a local ring scheme over a *quasi-compact* scheme S. In this section we will generalize Greenberg's notion of realization to the situation where X is a scheme over $R(S)[1/a]$, for $a \in R(S)$. Localized Greenberg realizations will be objects in the category of S-ind-schemes. Again, we remind the reader that the situation of interest for us will be the case where $S = \operatorname{Spec} k$ is the spectrum of a perfect field and $R = \mathrm{W}$ is the scheme of Witt vectors over S, and $a = p$ is a uniformizer.

Observe that the ring $R(S)[1/a]$ is the colimit of the inductive system of rings

$$R(S) \xrightarrow{\cdot a} R(S) \xrightarrow{\cdot a} R(S) \xrightarrow{\cdot a} \dots .$$

Assume again that R is isomorphic as an S-scheme to $\mathbb{A}^N_S$, i.e. that the affine line over $R(S)$ can be realized in the sense of Greenberg, Definition 3.2, by $\mathbb{A}^N_S$. Passing to Greenberg realizations we obtain the inductive system

$$\mathbb{A}^N_S \xrightarrow{F(\cdot a)} \mathbb{A}^N_S \xrightarrow{F(\cdot a)} \mathbb{A}^N_S \xrightarrow{F(\cdot a)} \dots .$$

If we denote the corresponding S-ind-scheme by $F_a \mathbb{A}^1_{R(S)}$, then for any S-scheme Y we obtain natural bijections

$$\begin{aligned} &\mathrm{Hom}_{S-ind-sch}(Y, F_a\mathbb{A}^1_{R(S)}) \simeq \varinjlim(\mathbb{A}^N_S(Y)) = \\ =&\varinjlim \mathrm{Hom}_{l.r.sp.}(G(Y), \mathbb{A}^1_{R(S)}) = \varinjlim R(Y) = R(Y)[1/a]. \end{aligned}$$

In other words, the functor $Y \mapsto R(Y)[1/a]$ is represented by the S-ind-scheme $F_a\mathbb{A}^1_{R(S)}$. This motivates the following definition.

Definition 3.7. Let X be an $R(S)[1/a]$-scheme. A *localized Greenberg realization* of X over S is an S-ind-scheme which represents the functor $Y \mapsto X(R(Y)[1/a])$ on the category of (quasi-compact) S-schemes.

Since the category of ind-schemes has fiber products, and by the universal property of Greenberg realizations, we obtain:

(1) Let $X \to T$ and $X' \to T$ be morphisms of $R(S)[1/a]$-schemes which admit localized Greenberg realizations (F_iX), (F_iX') and (F_iT) over S. Then the fiber product $(F_iX \times_{F_iT} F_iX')$ is a localized Greenberg realization over S of $X \times_T X'$.
(2) If X is a group scheme over $R(S)[1/a]$ having a localized Greenberg realization (F_iX) over S, then (F_iX) is a group object in the category of ind-schemes over S.

Remark 3.8. One could think of defining localized Greenberg realizations in a similar way as (usual) Greenberg realizations: namely, by defining a 'localized' functor $G[1/a]$, which takes any Y to a locally ringed space $G[1/a]Y$ over $R(S)[1/a]$, with underlying topological space the space of Y and $\Gamma(U, G[1/a]Y) = R(U)[1/a]$ for all open subsets $U \subset Y$. However, such a functor $G[1/a]$ seems not to exist, even

in the case $S = \operatorname{Spec} k, R = \mathrm{W}_N$ and $a = p$. Namely, contrary to what Haboush claims in his paper [Hab05], neither the functor $U \mapsto \mathrm{W}(U)[1/p]$ nor the functor $U \mapsto \mathrm{W}(U^{p^{-\infty}})[1/p]$ define a structure of *locally* ringed space on the underlying space of Y. Take for instance $Y = \operatorname{Spec} k[T]$. Then the stalk at the prime $(T) \in Y$ of the sheaf which is given on distinguished open subsets of Y by

$$\operatorname{Spec} k[T][1/f] \mapsto \mathrm{W}(k[T][1/f])[1/p], \quad f \in k[T],$$

contains the Witt-vectors $(-T, 0, \dots)$ and $(T, 1, 0, \dots)$ which are both non-units. However, their sum is a unit in $\mathrm{W}(k[T])[1/p]$. Thus the stalk at (T) is not local. The same counterexample works for the second functor, which is the one considered by Haboush.

Let us gather a few observations which we will use to prove the existence of localized Greenberg realizations in certain cases. First note that the existence of a localized Greenberg realization of the affine line $\mathbb{A}^1_{R(S)}$ is already proven by our remarks before Definition 3.7. Now let X be any affine scheme of finite type over $R(S)[1/a]$ and fix a closed immersion $X \subset \mathbb{A}^d_{R(S)[1/a]}$. Let moreover

$$\varphi_n : \mathbb{A}^d_{R(S)[1/a]} \to \mathbb{A}^d_{R(S)[1/a]}$$

be the automorphism given by $T_i \mapsto a^n T_i$ for $i = 1, \dots, d$. This yields a diagram of the form

$$\begin{tikzcd} \cdots \arrow[r] & \mathbb{A}^d_{R(S)[1/a]} \arrow[r, "\varphi_1"] & \mathbb{A}^d_{R(S)[1/a]} \arrow[r, "\varphi_1"] & \cdots \\ \cdots \arrow[r] & \varphi_n(X) \arrow[r] \arrow[u] & \varphi_{n+1}(X) \arrow[r] \arrow[u] & \cdots \end{tikzcd}$$

where all the horizontal maps are isomorphisms of $R(S)[1/a]$-schemes. Then define X_n to be the scheme-theoretic image of $\varphi_n(X) \hookrightarrow \mathbb{A}^d_{R(S)[1/a]} \hookrightarrow \mathbb{A}^d_{R(S)}$. In terms of ideals this means: if $I \subset R(S)[1/a][T_1, \dots, T_d]$ is the defining ideal of $X \subset \mathbb{A}^d_{R(S)[1/a]}$, then $X_n \subset \mathbb{A}^d_{R(S)}$ is defined by the ideal $R(S)[T_1, \dots, T_d] \cap (I|_{T_i \mapsto a^{-n} T_i})$. We obtain the $R(S)$-ind-scheme $(X_n)_n$.

In the sequel we write for any $R(S)$-scheme Y:

$$Y[1/a] := Y \times_{\operatorname{Spec} R(S)} \operatorname{Spec} R(S)[1/a].$$

With this notation we have $X_n[1/a] \simeq \varphi_n(X) \simeq_{\varphi_n^{-1}} X$ for all $n \in \mathbb{N}$.

Lemma 3.9. *The $R(S)$-ind-scheme $(X_n)_n$ represents the functor*

$$L : Y \mapsto \operatorname{Hom}_{R(S)[1/a]}(Y[1/a], X)$$

on the category of (quasi-compact) $R(S)$-schemes.

Proof. A morphism of functors $\psi_n : X_n \to L$ is given by the functorial map

$$\begin{aligned} X_n(Y) = \operatorname{Hom}_{R(S)}(Y, X_n) \to \operatorname{Hom}_{R(S)[1/a]}&(Y[1/a], X_n[1/a]) \\ &\simeq_{\varphi_n^{-1}} \operatorname{Hom}_{R(S)[1/a]}(Y[1/a], X). \end{aligned}$$

Obviously the morphisms ψ_n are compatible, so we obtain a morphism of functors $\psi : (X_n)_n \to L$. Since every $Y[1/a]$-valued point P of X is given by a d-tuple p in

$$\Gamma(Y[1/a])^d = (\Gamma(Y) \otimes_{R(S)} R(S)[1/a])^d,$$

there exists some $n \in \mathbb{N}$ such that $a^n \cdot p \in \Gamma(Y)^d$ and thus $\varphi_n(P)$ extends to a Y-valued point of X_n. This shows that $\psi(Y)$ is surjective for every $Y/R(S)$. To

check injectivity, take $P, Q \in X_n(Y)$ such that P and Q have the same image in $L(Y)$. This means in particular, that the corresponding morphisms $P', Q' : Y[1/a] \to X_n[1/a] = \varphi_n(X)$ are equal, and consequently the respective $R(S)$-morphisms $P'', Q'' : Y[1/a] \to Y \to X_n$ are equal. But both P and Q are given by d-tuples p, q of sections in $\Gamma(Y)$, and for these the equality $P'' = Q''$ says that there exists an $m \in \mathbb{N}$ such that $a^m p = a^m q$. This means that the compositions

$$Y \xrightarrow{P,Q} X_n \xrightarrow{\varphi_m} X_{n+m}$$

coincide, whence a fortiori P and Q coincide as elements of $(X_n)_n(Y)$. $\square$

It is now easy to construct localized Greenberg realizations for affine $R(S)[1/a]$-schemes which are of finite type.

Proposition 3.10. *Let X be an affine scheme of finite type over $R(S)[1/a]$, and assume that R is isomorphic as an S-scheme to some affine space over S. Then there exists an S-ind-scheme which represents the functor $Y \mapsto X(R(Y)[1/a])$ on the category of (quasi-compact) S-schemes.*

Proof. Fix a closed immersion $X \subset \mathbb{A}^d_{R(S)[1/a]}$ and let $(X_n)_n$ be as above. Now apply Greenberg realization to the $R(S)$-schemes X_n and their transition maps. I claim that the resulting S-ind-scheme $(FX_n)_n$ has the desired form. Indeed, we have

$$\begin{aligned}\operatorname{Hom}(Y, (FX_n)_n) &= \varinjlim \operatorname{Hom}(Y, FX_n) = \\ &= \varinjlim \operatorname{Hom}_{R(S)}(R(Y), X_n) = \operatorname{Hom}(R(Y)[1/a], X),\end{aligned}$$

where the second equality is by definition of Greenberg realization, and the third one follows from the previous lemma. $\square$

Example 3.11. Let us illustrate this in our standard situation of Witt vectors of infinite length over a (perfect) field k. Let $X = \mathbb{A}^d_{\mathrm{W}(k)[1/p]}$. Then the k-ind scheme which is the localized Greenberg realization of X is given (up to isomorphism) by the inductive system

$$\operatorname{Spec} k[x_{i,j}; i = 1, \ldots, d; j = 0, 1, \ldots] \xrightarrow{\cdot p} \operatorname{Spec} k[x_{i,j}] \xrightarrow{\cdot p} \ldots,$$

where the transition maps $\cdot p$ are defined by $x_{i,j} \mapsto x^p_{i,j-1}$ (for $j = 1, \ldots, \infty$) and $x_{i,0} \mapsto 0$.

4. Generalized loop group constructions

4.1. **Construction of generalized loop groups.** From now on we will consider the following situation: Let $\mathfrak{D}$ be a local ring scheme over a field k such that $D = \mathfrak{D}(k)$ is a discrete valuation ring with uniformizer $u \in D$. Moreover we assume that $\mathfrak{D}$ is isomorphic to $\mathbb{A}^{\mathbb{N}}_k$ as a scheme over k. Typical special cases are

(1) the scheme of power series in one variable over k, and
(2) the scheme of Witt vectors over a perfect field k of positive characteristic.

By K we denote the fraction field of D. Moreover, we now return to usual practice and use the letter R to denote a ring, usually a k-algebra.

Let X be a scheme over $\operatorname{Spec} D$. The functors on the category of k-algebras

$$\mathrm{L}\,X : R \mapsto X(\mathfrak{D}(R)[1/u])$$

and

$$\mathrm{L}^+ X : R \mapsto X(\mathfrak{D}(R))$$

will be called the (generalized) loop space, resp. positive loop space, associated with X. By abuse of notation we will write $\mathrm{L}\, X = \mathrm{L}(X \times_{\operatorname{Spec} D} \operatorname{Spec} K)$ for the loop space associated with $X \times_{\operatorname{Spec} D} \operatorname{Spec} K$. Obviously there is a natural morphism of functors $\mathrm{L}^+ X \to \mathrm{L}\, X$. If in addition $X = G$ is a group scheme over D, then we call $\mathrm{L}\, G$ and $\mathrm{L}^+ G$ the (generalized) *loop group* and the (generalized) *positive loop group*, respectively, associated with G.

Note that if $\mathfrak{D}$ is the k-scheme of power series in one variable over k, we recover the usual notions of (formal) loop space, loop group etc., as described by Beauville and Laszlo, [BL94], Pappas and Rapoport, [PR08], and others.

The following proposition is an immediate consequence of our discussion on Greenberg realizations:

Proposition 4.1. *If X is an affine scheme over D then the functor $\mathrm{L}^+ X$ is representable by an affine scheme over k, namely the Greenberg realization over k of X. If X is affine and of finite type over K, then $\mathrm{L}\, X$ is representable by the localized Greenberg realization over k of X.*

In fact, in all situations that we are going to consider the affine scheme X comes together with a 'natural' embedding into some affine space: $\iota : X \subset \mathbb{A}^d_K$. With respect to this embedding, the construction of the localized Greenberg realization $\mathrm{L}\, X$ described in the preceeding section produces an explicit direct system $(F X_i)_{i \in \mathbb{N}}$ of k-schemes which represents $\mathrm{L}\, X$. We will refer to this direct system as the 'natural representation' of $\mathrm{L}\, X$. Explicitly, the i-th step of the natural representation of $\mathrm{L}\, X$ parametrizes the K-points of X whose coordinates (with respect to the embedding ι) have 'poles' of order at most i.

4.2. **Operations.** Let G be a linear algebraic group over $D = \mathfrak{D}(k)$ and fix a closed immersion $G \subset \mathrm{Gl}_{n,D} \subset \mathbb{A}^{n \times n}_D$. The natural action of G on $\mathbb{A}^n_D$ induces, by functoriality of L and L^+, a commutative diagram

$$\begin{tikzcd}
\mathrm{L}\, G \times_k \mathrm{L}\, \mathbb{A}^n_K \arrow[r] & \mathrm{L}\, \mathbb{A}^n_K \\
\mathrm{L}^+ G \times_k \mathrm{L}^+ \mathbb{A}^n_D \arrow[r] \arrow[u] & \mathrm{L}^+ \mathbb{A}^n_D \arrow[u]
\end{tikzcd} .$$

It is easy to describe the action in the upper line explicitly in terms of the natural representations of the loop spaces involved. In fact, this action is described by the compatible system of maps

$$F(\mathbb{A}^{n \times n}_D)_m \times_k F(\mathbb{A}^n_D)_{m'} \to F(\mathbb{A}^n_D)_{m+m'},$$

where each of these maps is nothing but the usual 'multiplication of a matrix and a vector'. More precisely one could say that it is the (usual) Greenberg realization of the map $\mathbb{A}^{n \times n}_D \times \mathbb{A}^n_D \to \mathbb{A}^n_D$ given by multiplication of matrix and vector. The indices $m, m', m+m'$ may be explained as follows: if M is a k-point of $F(\mathbb{A}^{n \times n}_D)_m$ and v is a k-point of $F(\mathbb{A}^n_D)_{m'}$, then these two objects represent the elements $u^{-m} M \in G(K)$ and $u^{-m'} v \in \mathbb{A}^n(K)$, respectively. Their product is $u^{-m-m'} M \cdot v \in \mathbb{A}^n(K)$, which is thus represented by the product $M \cdot v$ – viewed as a k-point of $F(\mathbb{A}^n_D)_{m+m'}$.

Let us look at the $\mathrm{L}\,G$-action which is thereby induced on sub-ind-schemes of $\mathrm{L}\,\mathbb{A}^n_K$: For any k-scheme T let $\mathcal{S}(T)$ be the set of ind-closed T-sub-ind-schemes of $\mathcal{A}_T := (\mathrm{L}\,\mathbb{A}^n_K) \times_k T$, i.e. $\mathcal{S}(T)$ is the set of classes of commutative diagrams

$$\begin{tikzcd} \cdots \arrow[r] & (\mathbb{A}^{\mathbb{N}}_T)^n \arrow[r] & (\mathbb{A}^{\mathbb{N}}_T)^n \arrow[r] & \cdots, \\ \cdots \arrow[r] & L_i \arrow[r] \arrow[u, hook] & L_{i+1} \arrow[r] \arrow[u, hook] & \cdots \end{tikzcd}$$

where the vertical maps are closed immersions. Clearly, the assignment $T \mapsto \mathcal{S}(T)$ defines a functor on the category of k-schemes. From the $\mathrm{L}\,G$-operation on $\mathrm{L}\,\mathbb{A}^n_K$ we obtain an $\mathrm{L}\,G$-operation (on the right) on $\mathcal{S}$ by pulling back T-sub-ind-schemes along the morphism $\mathcal{A}_T \to \mathcal{A}_T$ given by a T-valued point of $\mathrm{L}\,G$. More precisely, we consider the natural morphism of functors

$$\rho'(T) : \mathrm{L}\,G(T) \times_T \mathcal{S}(T) \to \mathcal{S}(T); \quad (g, (L_i)) \mapsto (L_i) \times_{\mathcal{A}_T, g} \mathcal{A}_T.$$

Combining with the inverse map $\mathrm{L}\,G \to \mathrm{L}\,G, g \mapsto g^{-1}$, we can make this into a left-operation, which we denote by

$$\rho : \mathrm{L}\,G \times_k \mathcal{S} \to \mathcal{S}.$$

The action ρ can be described explicitly as follows: Let $g \in \mathrm{L}\,G(T) \subset \mathrm{L}\,\mathbb{A}^{n\times n}_K(T)$ be represented by $M \in F(G)_m(T) \subset F(\mathbb{A}^{n\times n}_D)_m(T)$. Then the map $\mathcal{A}_T \to \mathcal{A}_T$ which is induced by g is represented by the system of maps

$$F(\mathbb{A}^{n\times n}_D)_{m'} \to F(\mathbb{A}^{n\times n}_D)_{m'+m}; \quad (x_{i,j})_{1\le i\le n} \mapsto M \cdot (x_{i,j})_{1\le i\le n}$$

for any $m' \in \mathbb{N}$. Closed T-subschemes are pulled back as usual by plugging the defining polynomials of this morphism into their equations.

4.3. **The quotient $\mathrm{L}\,G/\mathrm{L}^+\,G$.** We are going to construct the quotient $\mathrm{L}\,G/\mathrm{L}^+\,G$ by considering the 'standard lattice', a certain sub-ind-scheme of $\mathrm{L}\,\mathbb{A}^n_D$, which has $\mathrm{L}^+\,G$ as its stabilizer.

Definition 4.2. The standard lattice $\mathbb{S} \subset \mathrm{L}\,\mathbb{A}^n_D$ is the fpqc-sheaf image of the natural map $\mathrm{L}^+\,\mathbb{A}^n_D \to \mathrm{L}\,\mathbb{A}^n_K$.

Lemma 4.3. *The standard lattice $\mathbb{S} \subset \mathrm{L}\,\mathbb{A}^n_K$ over k is the k-sub-ind-scheme represented by the diagram*

$$\begin{tikzcd} \cdots \arrow[r] & F\mathbb{A}^n_D \arrow[r, "F(\cdot u)"] & F\mathbb{A}^n_D \arrow[r] & \cdots \\ \cdots \arrow[r] & \mathbb{S}_i \arrow[r] \arrow[u, hook] & \mathbb{S}_{i+1} \arrow[r] \arrow[u, hook] & \cdots, \end{tikzcd}$$

where $\mathbb{S}_i$ is the scheme-theoretic image of $F\mathbb{A}^n_D = (\mathbb{A}^{\mathbb{N}}_k)^n$ under $F(\cdot u^i) = (F(\cdot u))^i$.

Proof. This is obvious. □

Let us consider the two standard situations for which these constructions are significant: (1) Let $D = k[[z]]$ be the power series ring in the variable z over k. Then for every k-algebra R our constructions yield:

$$\mathrm{L}^+\,G(R) = G(R[[z]]), \quad \mathrm{L}\,G(R) = G(R((z))),$$
$$\mathrm{L}^+\,\mathbb{A}^n_D(R) = \mathbb{S}(R) = R[[z]]^n, \quad \mathrm{L}\,\mathbb{A}^n_D(R) = R((z))^n.$$

(2) If $\mathfrak{D} = \mathrm{W}$ is the scheme of Witt vectors over k, then the situation is slightly more complicated, regarding the standard lattice: We have, for any k-algebra R,

$$\mathrm{L}^+ G(R) = G(\mathrm{W}(R)), \quad \mathrm{L}\, G(R) = G(\mathrm{W}(R)[1/p]),$$
$$\mathrm{L}^+ \mathbb{A}^n_D(R) = \mathrm{W}(R)^n, \quad \mathrm{L}\, \mathbb{A}^n_D(R) = \mathrm{W}(R)[1/p]^n.$$

The standard lattice $\mathbb{S}$ in this case is not the same as $\mathrm{L}^+ \mathbb{A}^n_D$: namely, multiplication by p in $\mathrm{W}(R)$ involves p-th roots of elements of R, which has the effect that the presheaf-image of $\mathrm{L}^+ \mathbb{A}^n_D$ is not an fpqc-sheaf and sheafification really produces a different object $\mathbb{S} \neq \mathrm{L}^+ \mathbb{A}^n_D$. For example, if $n = 1$ and $R = k[T]$, then $(T^{1/p}, 0, \dots)$ is not in $\mathrm{L}^+ \mathbb{A}^1_D(k[T])$. But it is in $\mathrm{L}^+ \mathbb{A}^1_D(k[T^{1/p}])$, and $k[T] \to k[T^{1/p}]$ is a faithfully flat extension. Thus $(T^{1/p}, 0, \dots) \in \mathbb{S}(R)$.

Theorem 4.4. *The stabilizer of the standard lattice under the action ρ is precisely the fpqc-sheaf-image of* $\mathrm{L}^+ G \to \mathrm{L}\, G$.

Proof. Let R be a k-algebra, and let $g \in \mathrm{L}\, G(R) = G(\mathfrak{D}(R)[1/u])$ be in the stabilizer of $\mathbb{S}(R) \subset \mathfrak{D}(R)[1/u]^n$. Consider the 'standard basis' $e_1, \dots, e_n \in \mathbb{S}(R)$. Then there exists a faithfully flat homomorphism of k-algebras $R \to R'$ such that the $g \cdot e_i$ induce R'-valued points of $\mathbb{S}$ which actually come from points in $\mathrm{L}^+ \mathbb{A}^n_D(R')$. In other words, g considered as an element of $G(\mathfrak{D}(R')[1/u])$ stabilizes $\mathfrak{D}(R')^n$, which shows that indeed $g \in G(\mathfrak{D}(R')) = \mathrm{L}^+ G(R')$. On the other hand, clearly every R-point of the sheaf-image of $\mathrm{L}^+ G \to \mathrm{L}\, G$ stabilizes the standard lattice. $\square$

5. The p-adic Grassmannian for Sl_n

In this section we apply the considerations of the previous paragraphs to the case where $G = \mathrm{Sl}_n$, the special linear group over a perfect field k, and where $\mathfrak{D} = \mathrm{W}$ is the scheme of Witt vectors over k, $u = p \in W(k)$. To make this situation also visible in our notation, we will henceforth write $\mathrm{L}_p\, G$ (instead of $\mathrm{L}\, G$) for the p-adic loop group, $\mathrm{L}^+_p\, G$ (instead of $\mathrm{L}^+ G$) for the positive p-adic loop group etc.

Denote by T the standard maximal torus contained in the standard Borel subgroup $B \subset \mathrm{Sl}_n$ of upper triangular matrices, and let $\check{\mathrm{X}}(T)$ and $\check{\mathrm{X}}_+(T)$ be the sets of cocharacters and dominant cocharacters, respectively. Identify $\check{\mathrm{X}}(T)$ with the subset of $\mathbb{Z}^n$ consisting of those vectors whose coordinates sum up to 0. Then $\check{\mathrm{X}}_+(T) \subset \mathbb{Z}^n$ consists of the vectors whose coordinates *decrease* and sum up to 0. Consider furthermore the embedding

$$\iota' : \check{\mathrm{X}}(T) \hookrightarrow G(K); \quad \lambda = (\lambda_1, \dots, \lambda_n) \mapsto \mathrm{diag}(p^{\lambda_1}, \dots, p^{\lambda_n}).$$

This embedding determines an embedding of $\check{\mathrm{X}}(T)$ into the loop group of G,

$$\iota : \check{\mathrm{X}}(T) \hookrightarrow \mathrm{L}_p\, G(k).$$

Hence, we may associate to every $\lambda \in \check{\mathrm{X}}_+(T)$ the k-sub-ind-scheme

$$\mathbb{S}_\lambda = \iota(\lambda) \cdot \mathbb{S} \in \mathcal{S}(k),$$

which can be explicitly described as follows: Let $\tilde{\lambda} = (\lambda_1 - \lambda_n, \dots, \lambda_{n-1} - \lambda_n, 0) \in \mathbb{N}^n$ and consider the closed sub-scheme of $F(\mathbb{A}^n_K)_{-\lambda_n}$ which is defined by the ideal

$$I_\lambda = \langle x_{1,0}, \dots, x_{1,\tilde{\lambda}_1 - 1}, \dots, x_{n-1,0}, \dots, x_{n-1,\tilde{\lambda}_{n-1}-1} \rangle.$$

Then $\mathbb{S}_\lambda$ is the fpqc-sheaf-image of the obvious map of this scheme into $\mathrm{L}_p\, \mathbb{A}^n_K$. Clearly, with this notation we have $\mathbb{S} = \mathbb{S}_0$.

Definition 5.1. The *p-adic Grassmannian* $\mathcal{G}rass_p$ is by definition the fpqc-sheaf-image of the map $\mathrm{L}_p\, G \to \mathcal{S}$, given by operation on the standard lattice. Moreover, the *Schubert cell* $\mathcal{C}_\lambda \subset \mathcal{G}rass_p$ is by definition the fpqc-sheaf-image of the map $\mathrm{L}_p^+\, G \to \mathcal{S}$ given by operation of $\mathrm{L}_p^+\, G$ on the sub-ind-scheme $\mathbb{S}_\lambda$.

Remark 5.2. In general the process of fpqc-sheafification presents set-theoretical problems, with the consequence that in certain cases one cannot speak of such sheafifications without making further restrictions (i.e. specifying a universe one wants to work in, causing the sheafification to depend on this choice). In the appendix to this paper we present an argument in order to prove that these problems do not occur in the present situation.

The k-valued points of the p-adic affine Grassmannian have a similar description as in the function field case.

Proposition 5.3. *The set of k-valued points of the p-adic Grassmannian is in bijective correspondence with the set of lattices in $\mathrm{W}(k)^n$. This correspondence is given by*

$$\begin{aligned} f : \mathcal{G}rass_p(k) &\xrightarrow{\simeq} \{\mathcal{L} \subset \mathrm{W}(k)^n \mid \mathcal{L} \text{ a lattice}\} \\ L &\mapsto L(k), \end{aligned} \tag{5.1}$$

i.e. the k-ind-scheme L is mapped to its set of k-valued points.

Proof. Observe that the set of k-valued points of the standard lattice $\mathbb{S}$ is equal to $\mathrm{W}(k)^n \subset \mathrm{W}(k)[1/p]^n = \mathrm{L}_p\, \mathbb{A}^n_{\mathrm{W}(k)[1/p]}(k)$. Now let $L = g \cdot \mathbb{S} \in \mathcal{G}rass_p$, where $g \in \mathrm{L}_p\, \mathrm{Sl}_n(k) = \mathrm{Sl}_n(\mathrm{W}(k)[1/p])$. This means that L is the pullback of $\mathbb{S}$ under $g^{-1} : \mathrm{L}_p\, \mathbb{A}^n_{\mathrm{W}(k)[1/p]} \to \mathrm{L}_p\, \mathbb{A}^n_{\mathrm{W}(k)[1/p]}$. Then $L(k) \subset \mathrm{W}(k)[1/p]^n$ is mapped bijectively onto $\mathbb{S}(k) = \mathrm{W}(k)^n$ by g^{-1}. In other words, $L(k) = g \cdot \mathrm{W}(k)^n$, i.e. a lattice. Since every lattice $\mathcal{L} \subset \mathrm{W}(k)[1/p]^n$ has the form $g \cdot \mathrm{W}(k)^n$, the map f is clearly surjective. Finally, if two k-sub-ind-schemes $g \cdot \mathbb{S}$ and $h \cdot \mathbb{S}$ have the same set of k-valued points, then the matrix $h^{-1}g \in \mathrm{Sl}_n(\mathrm{W}(k)[1/p])$ must actually lie in the stabilizer of $\mathrm{W}(k)^n$, i.e. in $\mathrm{Sl}_n(\mathrm{W}(k))$. But this implies $(h^{-1}g) \cdot \mathbb{S} = \mathbb{S}$, or equivalently, $h \cdot \mathbb{S} = g \cdot \mathbb{S}$, which proves injectivity. $\square$

By the elementary divisors-theorem, we see that

$$\mathcal{G}rass_p(k) = \cup_{\lambda \in \check{\mathrm{X}}_+(T)} \mathcal{C}_\lambda(k).$$

We will see in the following sections that, for any $\lambda \in \check{\mathrm{X}}_+(T)$, $\mathcal{C}_\lambda$ is representable by a quasi-projective k-scheme C_λ. Moreover, this affine k-scheme comes together with an open embedding into a projective k-scheme D_λ which maps naturally (as an fpqc-sheaf) to $\mathcal{G}rass_p$, thereby inducing the isomorphism $C_\lambda \simeq \mathcal{C}_\lambda$ as well as a surjection $D_\lambda(k) \twoheadrightarrow \cup_{\lambda' \leq \lambda} \mathcal{C}_{\lambda'}(k)$.

6. Hilbert schemes and lattice schemes

6.1. **Multigraded Hilbert schemes à la Haiman and Sturmfels.** We first recall a result by Haiman and Sturmfels ([HS04]) on the representability of the multigraded Hilbert functor.

Let R be any ring, and let $\mathbb{A}^n_R = \operatorname{Spec} R[x_1, \dots, x_n]$ be the n-dimensional affine space over R, and identify $u \in \mathbb{N}^n$ with the monomial $x_1^{u_1} \cdots x_n^{u_n}$. Then a multigrading of $R[x_1, \dots, x_n]$ by a semigroup A is given by a semigroup homomorphism

$\deg : \mathbb{N}^n \to A$. This induces a decomposition

$$R[x_1, \dots, x_n] = \oplus_{a \in A} R[x_1, \dots, x_n]_a,$$

where $R[x_1, \dots, x_n]_a$ is the R-span of the monomials of degree a.

A homogeneous ideal $I \subset R[x_1, \dots, x_n]$ is called an *admissible ideal*, if for each $a \in A$ the graded piece $(R[x_1, \dots, x_n]/I)_a$ is a locally free module of constant finite rank on $\operatorname{Spec} R$. Every admissible ideal $I \subset R[x_1, \dots, x_n]$ has then a well-defined *Hilbert function*, given by

$$h_I : A \to \mathbb{N}, \quad a \mapsto \operatorname{rk}(R[x_1, \dots, x_n]/I)_a.$$

An R-subscheme $V \subset \operatorname{Spec} R[x_1, \dots, x_n]$ which is defined by an admissible ideal will also be called admissible, and by the Hilbert function of such a V we will mean the Hilbert function of its defining ideal.

Let $h : A \to \mathbb{N}$ be any function supported on $\deg(\mathbb{N}^n)$, and define the *Hilbert functor* $\mathcal{H}^h_R : (R\text{-Alg}) \to (\text{Set})$ by

$$\mathcal{H}^h_R(S) = \{\text{admissible ideals } I \subset S[x_1, \dots, x_n] \mid \\ \operatorname{rk}(S[x_1, \dots, x_n]/I) = h(a) \text{ for all } a \in A\}.$$

Theorem 6.1 (Haiman, Sturmfels)**.** *There exists a quasiprojective scheme H^h_R over R which represents the functor $\mathcal{H}^h_R$. If the grading of $R[x_1, \dots, x_n]$ is positive, i. e. 1 is the only monomial with degree 0, then this scheme is even projective over R.* $\square$

The scheme H^h_R is called the 'multigraded Hilbert scheme' for the Hilbert function h. In the sequel, if we do not specify a Hilbert function h, then by the term 'multigraded Hilbert scheme', or just Hilbert scheme, we refer to the union of the multigraded Hilbert schemes for all possible Hilbert functions. We denote this scheme by H_R, or simply by H if the ring R is fixed.

6.2. **Lattice schemes.** For any ring scheme X over R we have the obvious notion of an X-module scheme over R. In particular, we have the free X-module scheme of rank n, denoted X^n. X-submodule schemes of an X-module scheme M are R-subschemes of M which are 'stable under the morphisms defining the module operations on M'. This means that for a closed X-subscheme $V \subset M$ we require the following diagrams to exist:

$$\begin{tikzcd} M \times M \arrow[r, "add."] & M \\ V \times V \arrow[u, hook] \arrow[r, dotted] & V \arrow[u, hook] \end{tikzcd} \qquad \begin{tikzcd} X \times M \arrow[r, "mult."] & M \\ X \times V \arrow[u, hook] \arrow[r, dotted] & V \arrow[u, hook] \end{tikzcd}$$

Analogous diagrams are required to exist for the zero-section and additive inverses.

In the sequel, we always assume that X is a ring scheme which is isomorphic as an R-scheme to $\mathbb{A}^N_R$ $(0 \le N < \infty)$. Let us furthermore fix a grading over R of the structure sheaf of $X \simeq \mathbb{A}^N_R$ so that the ring operations on X are defined by graded homomorphisms on the structure sheaf. Then also the structure sheaf of X^n is graded. We call a submodule scheme in X^n a *lattice-scheme* if its defining ideal is admissible.

Proposition 6.2. *The set of lattice schemes in X^n with given Hilbert function h is parametrized by a closed subscheme Z of the multigraded Hilbert scheme of X^n over R. The R-scheme Z is quasi-projective, and it is projective over R if the grading of X is positive.*

Proof. Let $H \to \operatorname{Spec} R$ be the multigraded Hilbert scheme of X^n and let $U \to H$ be the universal family. We have to show that there exists a closed subscheme $Z \subset H$ such that for any morphism $Y \to H$, $V = Y \times_H U \subset Y \times_{\operatorname{Spec} R} X^n$ is a submodule scheme if and only if $Y \to H$ factors through $Z \subset H$. It suffices to check this locally, i.e. for an affine open subscheme $H' = \operatorname{Spec} S \subset H$ instead of H itself. Then also $U' := H' \times_H U$ is affine, and U' is given by an ideal $I \subset S[x_{i,j} \mid i = 1, \dots, n; j = 0, 1, \dots, N]$ with S-locally free quotient $S[x_{i,j}]/I$. Now for any morphism $Y' = \operatorname{Spec} S' \to H'$ the condition that $V' = Y' \times_{H'} U' \subset U'$ be stable under the module operations on X^n translates into the condition that the image of I under the comorphism of addition vanishes in $S'[x_{i,j}]/I \otimes_{S'} S'[x_{i,j}]/I$, besides analogous vanishing conditions concerning scalar multiplication, units and additive inverses. Since $S[x_{i,j}]/I$ is locally free over S, these vanishing conditions can be expressed by equations with coefficients in S, which then define a closed subscheme $Z' \subset H' = \operatorname{Spec} S$. By construction, V' is stable under the module operations if and only if $Y' \to H'$ factors throuth Z'. By gluing all the $Z' \subset H$ we obtain the closed subscheme $Z \subset H$ which possesses the desired universal property. $\square$

Proposition 6.3 (Group actions on H)**.** *Let $\Gamma/\operatorname{Spec} R$ be an algebraic group acting algebraically on X^n, and assume that this action respects the grading on the structure sheaf of X^n. Then Γ acts on the Hilbert scheme H^h of X^n for any Hilbert function h. If furthermore the action of Γ on X^n is by automorphisms of X-module schemes, then the action of Γ on H^h restricts to an action on $Z^h = Z \cap H^h$.*

Proof. This is a formal consequence of the universal properties of H^h and Z^h and the fact that the action of Γ on X^n is algebraic, i.e. functorial. $\square$

6.3. **Lattice schemes in the Witt vector setting.** Let us specialize the above discussion to the case where

$$X = \mathrm{W}_N = \operatorname{Spec} k[\alpha_0, \dots, \alpha_{N-1}]$$

is the scheme of Witt-vectors over $S = \operatorname{Spec} k$ of length $N < \infty$, k being a perfect field of positive characteristic p, $D = \mathrm{W}(k)$, K its quotient field and $u = p \in D$. Further, we restrict to the case $G = \mathrm{Sl}_n$. This will be the setting to work with in the next sections.

For any $n \in \mathbb{N}$ we consider

$$\mathrm{W}_N^n \simeq \mathbb{A}_k^{N \times n} = \operatorname{Spec} k[x_{i,j} \mid i = 1, \dots, n; j = 0, \dots, N-1].$$

This can be seen as the Greenberg realization of $\mathrm{W}_N(k)^n$, and by functoriality of Greenberg realization we obtain the obvious structure of W_N-module scheme on W_N^n. The morphisms defining the module operations on the scheme W_N^n are defined by *graded* homomorphisms of the affine rings as soon as we fix the grading

$$\deg x_{i,j} = p^j, \quad x_{i,j} \in \Gamma(\mathrm{W}_N^n, \mathcal{O}), \tag{6.1}$$

and analogously, $\deg \alpha_j = p^j$. This follows from the definition of Witt vector arithmetics. Note that the standard grading $\deg x_{i,j} = 1$ is not respected by the module-operations and is hence not suited for our constructions.

Let us look at lattice schemes inside W_N^n. For any $\lambda = (\lambda_1 \geq \ldots \geq \lambda_n) \in \mathbb{Z}^n$ set $\tilde{\lambda} = (\lambda_1 - \lambda_n, \ldots, \lambda_{n-1} - \lambda_n)$ and define the ideal

$$I_\lambda = \langle x_{1,0}, \ldots, x_{1,\tilde{\lambda}_1 - 1}, \ldots, x_{n-1,0}, \ldots, x_{n-1,\tilde{\lambda}_{n-1}-1} \rangle.$$

Choose $N > \tilde{\lambda}_1$. Then this ideal determines a lattice scheme $V_\lambda \subset (\mathbb{A}_k^N)^n$. We denote by C_λ the orbit of $V_\lambda \in H$ under the action of the linear k-group $FG = \mathrm{L}^+ G$ on H, and by D_λ its orbit-closure in H. Theorem 6.1 asserts in particular that D_λ is a projective k-variety, which contains C_λ as an open subvariety. The variety C_λ will turn out to represent the Schubert cell $\mathcal{C}_\lambda \subset \mathcal{G}rass_p$.

7. Morphisms from D_λ to the p-adic affine Grassmannian

7.1. **Construction of a morphism $D_\lambda \to \mathcal{G}rass_p$.** The purpose of this section is to relate the constructions of the two preceding sections, i.e. the construction of the p-adic affine Grassmannian on the one hand, and the orbit-closure $D_\lambda \subset H$ on the other hand, by a morphism of fpqc-sheaves

$$D_\lambda \to \mathcal{G}rass_p.$$

Fix $\lambda \in \check{\mathrm{X}}_+(T) \subset \mathbb{Z}^n$ and let $D_\lambda \subset H$ be the orbit-closure constructed in Section 6.3. Let moreover U_λ be the universal family obtained by pull-back from the universal family over H:

$$\begin{tikzcd} U_\lambda \arrow[r] \arrow[d] & U \arrow[d] \\ D_\lambda \arrow[r] & H. \end{tikzcd}$$

We consider the determinant

$$\det : (\mathrm{W}_N(k)^n)^n = \mathrm{W}_N(k)^{n \times n} \to \mathrm{W}_N(k)$$

and its Greenberg realization

$$\Delta = (\Delta_0, \ldots, \Delta_{N-1}) = F(\det) : (\mathbb{A}_k^{N \times n})^n \to \mathbb{A}_k^N.$$

We obtain the commutative diagram

$$\begin{tikzcd} (U_\lambda)^n \arrow[r, hook] \arrow[dr] & (\mathbb{A}_{D_\lambda}^{N \times n})^n \arrow[r, "\Delta"] \arrow[d] & \mathbb{A}_{D_\lambda}^N \arrow[dl] \\ & D_\lambda & \end{tikzcd}.$$

Let us set $\Lambda = -n\lambda_n (= \tilde{\lambda}_1 + \cdots + \tilde{\lambda}_n)$ and observe, that in fact we have a factorization

$$\Delta : (U_\lambda)^n \to 0 \times \cdots \times 0 \times \mathbb{A}_{D_\lambda}^{N-\Lambda} \hookrightarrow \mathbb{A}_{D_\lambda}^N,$$

which can be interpreted as follows: The set of k-valued points of the fiber in U_λ over any point in D_λ is a submodule of $\mathrm{W}_N(k)^n$ whose determinant is precisely the ideal $(p^\Lambda) \subset \mathrm{W}_N(k)$. This factorization is due to the fact that Δ factorizes in this way over the point $V_\lambda \in D_\lambda$ and hence over its $\mathrm{L}_p^+ \mathrm{Sl}_n$ orbit, since the determinant map is invariant under the $\mathrm{L}_p^+ \mathrm{Sl}_n$-operation on $U_\lambda \to D_\lambda$. Consequently, Δ factorizes also over the orbit-closure D_λ.

We set $Y := 0 \times \cdots \times 0 \times \mathbb{A}_{D_\lambda}^{N-\Lambda}$, and further we denote by $Y' \subset Y$ the open subvariety $Y' = 0 \times \cdots \times 0 \times (\mathbb{A}_{D_\lambda} - \{0\}) \times \mathbb{A}_{D_\lambda}^{N-\Lambda-1}$. While the k-valued points of Y correspond to the Witt vectors in $(p^\Lambda) \subset \mathrm{W}_N(k)$, the k-valued points of Y' correspond to $(p^\Lambda) - (p^{\Lambda+1}) \subset \mathrm{W}_N(k)$.

We define X so to make the following diagram cartesian:

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y' \arrow[d, "\varphi"] \\ (U_\lambda)^n \arrow[r, "\Delta"] & Y. \end{tikzcd}$$

Thus X is an open subvariety of $(U_\lambda)^n$, and since $U_\lambda \to D_\lambda$ is flat by construction, also the morphism $X \to D_\lambda$ is *flat*. We even have

Proposition 7.1. *The morphism $X \to D_\lambda$ is faithfully flat and quasi-compact.*

Proof. Quasi-compactness of $X \to D_\lambda$ is trivial, and we have already argued that it is flat. So it remains to check its surjectivity. But this is also easy: Take any point $x \in D_\lambda$, and let $\kappa(x)$ be its residue field and $\overline{\kappa(x)}$ its algebraic closure. Then the fiber $(U_\lambda)^n \times_{D_\lambda} \overline{\kappa(x)}$ admits a family of $\overline{\kappa(x)}$-valued points with determinant $\equiv p^\Lambda \mod p^{\Lambda+1}$, i.e. a section which factors through the subscheme $X \subset (U_\lambda)^n$. □

This gives us an fpqc-covering $X \to D_\lambda$ with the property that 'locally on X' the family $(U_\lambda)^n \to D_\lambda$ has an algebraic family of sections $s : X \to (U_\lambda)^n \times_{D_\lambda} X$ whose determinant is non-zero $\mod p^{\Lambda+1}$. Namely, we can take

$$\begin{tikzcd} X \times_{D_\lambda} (U_\lambda)^n \arrow[r] \arrow[d] & (U_\lambda)^n \arrow[d] \\ X \arrow[r] \arrow[u, bend left, "s"] & D_\lambda, \end{tikzcd}$$

the section s on the left being given by the product of the identity and the open immersion $X \hookrightarrow (U_\lambda)^n$. In other words, this family provides, when pulled back to any $x \in X$, a basis of the free $\mathrm{W}_N(\overline{\kappa(x)})$-module $U_\lambda(\overline{\kappa(x)})$. We will use this section s to give an $\mathrm{L}_p^+ \mathrm{Sl}_n$-equivariant morphism

$$X \to \mathrm{L}_p \mathrm{Sl}_n \to \mathcal{G}rass_p.$$

To this end we consider the following closed embedding of Greenberg realizations:

$$F(\mathrm{Mat}_{\mathrm{W}_N(k)}(n \times n)) \simeq (\mathbb{A}_k^N)^{n\times n} \hookrightarrow (\mathbb{A}_k^\infty)^{n\times n} = F(\mathrm{Mat}_{\mathrm{W}(k)}(n \times n)), \tag{7.1}$$

given by $(x_{(i,l),j=0,\dots,N-1}) \mapsto (x_{(i,l),0}, \dots, x_{(i,l),N-1}, 0, 0, \dots)$. It induces a map

$$X \to F(\mathrm{Gl}_{n,K})_{-\lambda_n} \subset F(\mathrm{Mat}_{n,K})_{-\lambda_n} = (\mathbb{A}_k^\infty)^{n\times n},$$

($F(\cdot)_{-\lambda_n}$, as usual, denotes the $-\lambda_n$-th scheme in the natural representation of the respective k-ind-schemes) and thus a morphism $X \to \mathrm{L}_p \mathrm{Gl}_{n,K}$. Composing with the morphism $\mathrm{L}_p(\mathrm{Gl}_{n,K} \to \mathrm{Sl}_{n,K})$ which divides the first column of any invertible matrix by its determinant, we obtain a morphism of k-ind-schemes

$$\Phi : X \to \mathrm{L}_p \mathrm{Sl}_n,$$

and hence $\bar\Phi : X \to \mathcal{G}rass_p$. In order to show that this morphism is $\mathrm{L}_p^+ \mathrm{Sl}_n$-equivariant we have to check that $\bar\Phi$ does not 'depend on the 0's' in the map in (7.1), or, in other words, that putting any other sections of X in place of the 0's in (7.1) would not change $\bar\Phi$. This will follow from

Lemma 7.2. *Let us agree for the formulation and proof of this lemma that by $\mathrm{Sl}_n(\mathrm{W}(R))$ we mean the image of the morphism $\mathrm{Sl}_n(\mathrm{W}(R)) \to \mathrm{Sl}_n(\mathrm{W}(R)[1/p])$ (which is not the same if R is non-reduced), and analogously for $\mathrm{Mat}_n(\mathrm{W}(R))$.*

Let $A \in \mathrm{Sl}_n(\mathrm{W}(R)[1/p])$ such that $p^{\lambda_1}A^{-1} \in \mathrm{Mat}_n(\mathrm{W}(R))$. If $B \in \mathrm{Sl}_n(\mathrm{W}(R)[1/p])$ with $A - B \in p^{\lambda_1+1}\,\mathrm{Mat}_n(\mathrm{W}(R))$, then $A^{-1}B \in \mathrm{Sl}_n(\mathrm{W}(R))$.

Proof. We have $1 - A^{-1}B = A^{-1}(A-B) \in p\,\mathrm{Mat}_n(\mathrm{W}(R))$. Using the geometric series one sees that $A^{-1}B$ is invertible in $\mathrm{Mat}_n(\mathrm{W}(R))$, i.e. is an element of $\mathrm{Gl}_n(\mathrm{W}(R))$. As both A and B have determinant 1, so has $A^{-1}B$. □

In particular, A and B as in the lemma induce the same morphism $R \to \mathrm{L}_p\,\mathrm{Sl}_n / \mathrm{L}_p^+\,\mathrm{Sl}_n \to \mathcal{G}rass_p$. Now recall that we chose $N > \tilde{\lambda}_1 = \lambda_1 - \lambda_n$. Thus changing the morphism of (7.1) in those coordinates with $j \geq N$ amounts to changing the morphism Φ by something in $p^{\lambda_1+1}\,\mathrm{Mat}_2(\mathrm{W}(R))$. So the lemma tells us that in any case we get the same $\bar{\Phi} : X \to \mathcal{G}rass_p$, and $X \to \mathcal{G}rass_p$ is thus $\mathrm{L}_p^+\,\mathrm{Sl}_n$-equivariant.

By $\mathcal{X} \in \mathcal{G}rass_p(X)$ we denote the X-valued point corresponding to $\bar{\Phi}$.

Theorem 7.3. *The X-valued point $\mathcal{X} \in \mathcal{G}rass_p(X)$ descends to a D_λ-valued point of $\mathcal{G}rass_p$. The corresponding morphism $\varphi : D_\lambda \to \mathcal{G}rass_p$ is equivariant for the (left-)action of $\mathrm{L}^+\,\mathrm{Sl}_n$ and sends the lattice scheme V_λ to the lattice $\mathrm{diag}(p^{\lambda_1}, \ldots, p^{\lambda_n}) \cdot \mathbb{S} = \mathbb{S}_\lambda$. Moreover, this map restricts to an isomorphism of the respective Schubert cells: $C_\lambda \simeq \mathcal{C}_\lambda$.*

Proof. Since by definition $\mathcal{G}rass_p$ is an fpqc-sheaf and by Proposition 7.1 $X \to D_\lambda$ is faithfully flat, we have an exact sequence

$$\mathcal{G}rass_p(D_\lambda) \hookrightarrow \mathcal{G}rass_p(X) \rightrightarrows \mathcal{G}rass_p(X \times_{D_\lambda} X).$$

So we have to show that $\mathcal{X}$ is in the difference kernel of the maps $\mathcal{G}rass_p(X) \rightrightarrows \mathcal{G}rass_p(X \times_{D_\lambda} X)$. In order to see this, we compare the two composites

$$\Phi_1, \Phi_2 : X \times_{D_\lambda} X \rightrightarrows X \xrightarrow{\Phi} \mathrm{L}_p\,\mathrm{Sl}_n\,.$$

More precisely, we show that $\Phi_1^{-1} \cdot \Phi_2 \in \mathrm{L}_p\,\mathrm{Sl}_n(X \times_{D_\lambda} X)$ is in the stabilizer of the standard lattice $\mathbb{S} \times_k (X \times_{D_\lambda} X)$. By Theorem 4.4 this implies that fpqc-locally we have $\Phi_1^{-1} \cdot \Phi_2 \in \mathrm{L}^+\,\mathrm{Sl}_n$, which in turn shows that both images of $\mathcal{X}$ in $\mathcal{G}rass_p$ coincide. The point $\Phi_1^{-1} \cdot \Phi_2$ is represented by the matrix $\tilde{\Phi}_1^* \cdot \tilde{\Phi}_2 \in F(\mathrm{Sl}_{n,K})_\Lambda(X \times_{D_\lambda} X)$ (the $*$ denoting adjoint matrices). We may check that it maps the standard-lattice into itself on each finite piece of the standard lattice, and there (since we are dealing with k-varieties) we may look at fibers over closed points. But closed points of the product $X \times_{D_\lambda} X$ correspond to pairs of closed points of X in the same fiber over D_λ, which in turn correspond to two choices of basis of one and the same lattice. Thus the pull-back of $\tilde{\Phi}_1^* \cdot \tilde{\Phi}_2$ to any closed point of $X \times_{D_\lambda} X$ stabilizes the standard lattice, whence the same holds for $\tilde{\Phi}_1^* \cdot \tilde{\Phi}_2$ itself. Thus the map $\bar{\Phi} : X \to \mathcal{G}rass_p$ descends to D_λ – in other words, we obtain a factorization

$$\bar{\Phi} : X \to D_\lambda \xrightarrow{\varphi} \mathcal{G}rass_p.$$

Next we check that the k-valued point $V_\lambda \in D_\lambda$ maps to $\mathbb{S}_\lambda$ under φ. Obviously, it is sufficient to take any k-valued point in the fiber of $X \to D_\lambda$ over V_λ and calculate its image under $\bar{\Phi}$. By construction, the fiber of $X \to D_\lambda$ over V_λ is an open subset of the affine k-scheme $(V_\lambda)^n$, and contains a k-valued point $(P_1, \ldots, P_n)$ representing the vectors

$$(p^{\tilde{\lambda}_1}, 0, \ldots, 0)^T, \ldots, (0, \ldots, 0, p^{\tilde{\lambda}_n})^T \in \mathbb{A}^n_{\mathrm{W}_N(k)}.$$

Obviously, the matrix $P = (P_1, \ldots, P_n)$ is mapped to $\mathbb{S}_\lambda$ under $\bar{\Phi}$.

Moreover, the fact that $V_\lambda \in C_\lambda$ and $\mathbb{S}_\lambda \in \mathcal{S}(k)$ have the same stabilizer in $\mathrm{L}^+ \mathrm{Sl}_n(R)$, for any k-algebra R, shows that $D_\lambda \to \mathcal{G}rass_p$ restricts to an isomorphism $C_\lambda \simeq \mathcal{C}_\lambda$. □

7.2. **Properties of the morphism $D_\lambda \to \mathcal{G}rass_p$.** It would be desirable that the isomorphism $C_\lambda \simeq \mathcal{C}_\lambda$ extended to a closed immersion of functors $D_\lambda \to \mathcal{G}rass_p$, in order to obtain 'Schubert varieties' in the p-adic setting. Unfortunately, this is not the case, for the reason that the final assertion on the equality of stabilizers in the preceeding proof does not hold for points in $D_\lambda - C_\lambda$. For example, let $n = 2$ and let $\lambda = (-1, 1) \in \mathbb{Z}^2$, $N = 2$. Then $V_\lambda \subset \mathbb{A}_k^{2\times 2} = \operatorname{Spec} k[x_0, x_1, y_0, y_1]$ is defined by the ideal $\langle y_0, y_1\rangle$. Further, $D_\lambda - C_\lambda$ contains a whole $\mathbb{A}_k^1$, whose k-points $P_a, a \in k$, are given by ideals of the form $\langle x_0 + ay_0, y_0^p\rangle$. This whole $\mathbb{A}^1$ maps to the standard lattice $\mathbb{S} \in \mathcal{G}rass_p(k)$. Put otherwise, the standard lattice is fixed e.g. by the matrix which swaps the x- and y-coordinates, while the points P_a are not fixed.

There seems to be no way out of this situation. Namely, the reason for the phenomenon that there are in general many different points in D_λ mapping to the same point in $\mathcal{G}rass_p$ is the following: The subschemes of affine space which correspond to points in $D_\lambda - C_\lambda$ carry infinitesimal structure, which is forgotten by the map $D_\lambda \to \mathcal{G}rass_p$. On the other hand, these infinitesimal structures cannot be avoided as soon as we try to represent lattices by points in a Hilbert scheme, since we are then forced to use a Hilbert scheme for a non-standard grading as described in (6.1). E.g. in the before-mentioned example the ideals $\langle y_0, y_1\rangle$ and $\langle x_0, y_0\rangle$ can never have the same Hilbert function, whence the latter cannot lie in the orbit-closure in H of the former. However, e.g. $\langle x_0^p, y_0\rangle$ will be in the orbit-closure of $\langle y_0, y_1\rangle$.

An analogous situation can be constructed in the function field case, where the analogon of D_λ then turns out to be closely related to Demazure resolutions of Schubert varieties in the affine Grassmannian (see [Kre10] for details). This of course suggests to think of D_λ also in the present Witt vector setting as some sort of Demazure resolution of a Schubert variety in $\mathcal{G}rass_p$, but I do not know at present how to make this a precise statement.

Though we have seen that the morphism $D_\lambda \to \mathcal{G}rass_p$ is in general not injective on the level of k-valued points, its image behaves as expected.

Theorem 7.4. *We have $\mathcal{G}rass_p(k) = \cup_{\lambda \in \check{\mathrm{X}}_+(T)} \mathcal{C}_\lambda(k)$, and on the level of k-valued points, $D_\lambda \to \mathcal{G}rass_p$ induces a surjection*

$$\pi : D_\lambda(k) \twoheadrightarrow \cup_{\lambda' \leq \lambda} \mathcal{C}_{\lambda'}(k).$$

The symbol $\leq$ here refers to the Bruhat-order on $\check{\mathrm{X}}_+(T) \subset \mathbb{Z}^n$.

Proof. The first claim follows from the the elementary divisors theorem, as we have already explained at the end of section 5. In order to see that π is well-defined, we have to argue that none of the $\mathbb{S}_{\lambda'}$ with $\lambda' > \lambda \in \check{\mathrm{X}}_+(T)$ is in the image of π. So choose a $\lambda' > \lambda$. Then an argument as in [Kre10], Lemma 6.9 and Corollary 6.10, shows that the Hilbert function of the lattice scheme $V_{\lambda'}$ is bigger than that of V_λ itself (where for two functions $h, h' : \mathbb{N} \to \mathbb{N}$ we say $h' > h$ if and only if $h'(n) > h(n)$ for all n). But since V_λ is reduced, it has already the smallest possible Hilbert function among those lattice schemes which possibly map to $\mathbb{S}_{\lambda'}$. As D_λ contains only lattice schemes with the same Hilbert function as V_λ, $V_{\lambda'} \notin D_\lambda(k)$.

In order to prove surjectivity of π, we use an argument similar to the one given by Beauville and Laszlo in [BL94], Proposition 2.6.: For integers $e > d$ consider the following equation of matrices over $\mathrm{W}(k((t)))[1/p]$.

$$\begin{pmatrix} 0 & t \\ -t^{-1} & t^{-1}p \end{pmatrix} \begin{pmatrix} p^e & 0 \\ 0 & p^d \end{pmatrix} \begin{pmatrix} t^{-1} & 0 \\ t^{-1}p^{e-d-1} & t \end{pmatrix} = \begin{pmatrix} p^{e-1} & t^2p^d \\ 0 & p^{d+1} \end{pmatrix}. \tag{7.2}$$

If we assume $e + d = 0$, it follows that the right hand matrix gives rise to a lattice scheme $V \in D_{(e,d)}(k((t)))$, which corresponds to a $k((t))$-point of $\mathcal{C}_{(e,d)}$. Since $D_{(d,e)}$ is projective, this $k((t))$-valued point extends to a lattice scheme $\bar{V}$ over $k[[z]]$, whose fiber over $t = 0$ maps to $\mathbb{S}_{(d+1,e-1)}$. The case for a general n and λ is proved likewise. □

8. Lattices over the Witt ring

In Section 5 we have described the set of k-valued points of the p-adic affine Grassmannian $\mathcal{G}rass_p$. The purpose of the present section is to describe the R-valued points of $\mathcal{G}rass_p$ for more general k-algebras R.

Let us remind the reader of the situation in the function field case: There we have the following definition and theorem (see Beauville and Laszlo, [BL94], or Görtz, [Gör10]).

Definition 8.1. Let R be a k-algebra. A *lattice* $L \subset R((z))^n$ is a finitely generated projective $R[[z]]$-submodule such that $L \otimes_{R[[z]]} R((z)) = R((z))^n$. A lattice L is called *special*, if its determinant is trivial, i.e. $\wedge^n L = R[[z]]$.

Theorem 8.2. *For an $R[[z]]$-submodule $L \subset R((z))^n$ the following are equivalent:*

1. *The submodule L is a lattice.*
2. *Zariski-locally on R, L is a free $R[[z]]$-submodule of rank n (i.e. there exist $f_1, \dots, f_r \in R$ such that $(f_1, \dots, f_r) = R[[z]]$ and for all i, $L \otimes_{R[[z]]} R_{f_i}[[z]]$ is free of rank n and $L \otimes_{R[[z]]} R((z)) = R((z))^n$.*
3. *fpqc-locally on R, L is a free $R[[z]]$-submodule of rank n (i.e. there exists a faithfully flat ring homomorphisms $R \to S$ such that $L \otimes_{R[[z]]} S[[z]]$ is free of rank n and $L \otimes_{R[[z]]} R((z)) = R((z))^n$.*
4. *There exists a positive integer N such that $z^N R[[z]]^n \subset L \subset z^{-N} R[[z]]^n$ and $z^{-N} R[[z]]^n / L$ is a projective R-module.*

Our goal is to obtain a similar result in the Witt vector setting in the case where R is a perfect k-algebra. As a corollary we will then obtain a description in terms of 'lattices' of the R-valued points of $\mathcal{G}rass_p$ for R perfect. Recall that a ring R of characteristic $p > 0$ is called perfect, if the Frobenius homomorphism $x \mapsto x^p$ is an isomorphism.

Definition 8.3. Let R be a perfect ring. A *lattice* $L \subset \mathrm{W}(R)[1/p]^n$ (or simply: a $\mathrm{W}(R)$-lattice of rank n) is a finitely generated, projective $\mathrm{W}(R)$-submodule $L \subset \mathrm{W}(R)[1/p]^n$ such that $L \otimes_{\mathrm{W}(R)} \mathrm{W}(R)[1/p] = \mathrm{W}(R)[1/p]^n$. Further, a lattice $L \subset \mathrm{W}(R)$ is called *special*, if $\wedge^n L = \mathrm{W}(R)$. By $\mathcal{L}att_p^n(R)$ we denote the set of lattices of rank n over $\mathrm{W}(R)$, and $\mathcal{L}att_p^{n,0}(R) \subset \mathcal{L}att_p^n(R)$ is the subset of special lattices.

If $R = k$ is a field, then we recover the usual notion of lattice over $\mathrm{W}(k)$. Let us note furthermore that for a finitely generated $\mathrm{W}(R)$-submodule $L \subset \mathrm{W}(R)^n$

the condition $L \otimes_{\mathrm{W}(R)} \mathrm{W}(R)[1/p] = \mathrm{W}(R)[1/p]^n$ is equivalent to the existence of a natural number N such that $p^N \mathrm{W}(R)^n \subset L \subset p^{-N} \mathrm{W}(R)^n$.

First we want to see that the assignment $R \mapsto \mathcal{L}att_p^n(R)$ is a functor on the category of perfect k-algebras. To this end, we prove

Lemma 8.4. *Let $R \to S$ be a homomorphism of perfect rings, and let $p^N \mathrm{W}(R)^n \subset L \subset p^{-N} \mathrm{W}(R)^n$ be a flat $\mathrm{W}(R)$-submodule. Then we have*

$$\mathrm{Tor}_1^{W(R)}(\mathrm{W}(R)^r/L, \mathrm{W}(S)) = 0.$$

Equivalently, this means $L \otimes_{\mathrm{W}(R)} \mathrm{W}(S) \subset p^{-N} \mathrm{W}(S)^n \subset \mathrm{W}(S)[1/p]^n$.

Proof. Let $F = p^{-N} \mathrm{W}(R)^n$ and consider the exact sequence

$$0 \to \mathrm{Tor}_1^{W(R)}(F/L, \mathrm{W}(S)) \to L \otimes_{\mathrm{W}(R)} \mathrm{W}(S) \to \mathrm{W}(S)^n \to F/L \otimes_{\mathrm{W}(R)} \mathrm{W}(S) \to 0.$$

Since multiplication by p^{2N} is the zero-map on F/L, we see that p acts nilpotently on $\mathrm{Tor}_1^{W(R)}(F/L, \mathrm{W}(S))$. On the other hand, $(L \xrightarrow{p} L) \otimes \mathrm{W}(S) = L \otimes (\mathrm{W}(S) \xrightarrow{p} \mathrm{W}(S))$ is injective, since L is flat. Hence, p acts faithfully on $\mathrm{Tor}_1^{W(R)}(F/L, \mathrm{W}(S))$, which therefore vanishes. □

Proposition 8.5. *The assignment $R \mapsto \mathcal{L}att_p^n(R)$ defines a functor from the category of perfect rings to the category of sets. Namely, to any homomorphism $R \to S$ assign the map*

$$\mathcal{L}att_p^n(R) \to \mathcal{L}att_p^n(S); \quad L \mapsto L \otimes_{\mathrm{W}(R)} \mathrm{W}(S).$$

The assignment $R \mapsto \mathcal{L}att_p^{n,0}(R)$ is a subfunctor. □

The rest of this section is devoted to the study of the Zariski-/fpqc-sheaf properties of $\mathcal{L}att_p^n$ resp. $\mathcal{L}att_p^{n,0}$.

Theorem 8.6. *(1) The functor $\mathcal{L}att_p^n$ is the Zariski-sheafification of the functor on the category of perfect k-algebras, which associates to any perfect k-algebra R the set of* free *rank-n lattices over $\mathrm{W}(R)$.*

(2) Moreover, $\mathcal{L}att_p^n$ is even an fpqc-sheaf on the category of perfect k-algebras. Together with (1) this says that $\mathcal{L}att_p^n$ is also the fpqc-sheafification of the functor which associates to any perfect k-algebra R the set of free rank-n lattices over $\mathrm{W}(R)$.

(3) The analogous assertions hold if we replace $\mathcal{L}att_p^n$ by $\mathcal{L}att_p^{n,0}$ and 'free lattices' by 'free special lattices'.

Proof. It suffices to prove the first two parts of the theorem, part (3) will then follow. The first part of the theorem is easy: Since by definition $L \in \mathcal{L}att_p^n(R)$ is projective and finitely generated as a $\mathrm{W}(R)$-module, it is even finitely presented and (Zariski-)locally free over $\mathrm{W}(R)$. This means that there exist Witt vectors $f_1, \dots, f_m \in \mathrm{W}(R)$ which generate the unit ideal in $\mathrm{W}(R)$ and such that for each $1 \le i \le m$ the localization $L \otimes_{\mathrm{W}(R)} \mathrm{W}(R)[1/f_i]$ is free over $\mathrm{W}(R)[1/f_i]$. Denote by $g_i \in R$ the class mod p of f_i. Then the g_i generate the unit ideal in R, and I claim that the $\mathrm{W}(R[1/g_i])$-module $L \otimes_{\mathrm{W}(R)} \mathrm{W}(R[1/g_i])$ is free for each i. Namely, if we denote by $[g_i]$ the Teichmüller representative of g_i we may write

$$f_i = [g_i] \cdot \alpha, \quad \alpha \in 1 + p\,\mathrm{W}(R) \subset \mathrm{W}(R)^\times$$

since we assumed R to be perfect. Thus we have $\mathrm{W}(R)[1/f_i] = \mathrm{W}(R)[1/[g_i]] \subset \mathrm{W}(R[1/g_i])$. Hence, we may choose $\coprod_{i=1}^m \operatorname{Spec} R[1/g_i] \to \operatorname{Spec} R$ as a Zariski-covering on which L becomes free.

The proof of part (2) requires more work and will occupy us for the rest of this section.

Lemma 8.7. *Let $R \to S$ be a homomorphism of* perfect *rings. Then*

$$\mathrm{W}_N(S) \otimes_{\mathrm{W}_N(R)} \mathrm{W}_N(S) = \mathrm{W}_N(S \otimes_R S).$$

Proof. The ring $\mathrm{W}(S \otimes_R S)$ carries a natural structure of $\mathrm{W}(R)$-module, and for this module structure we have a linear map $\mathrm{W}(S) \otimes_{\mathrm{W}(R)} \mathrm{W}(S) \to \mathrm{W}(S \otimes_R S)$. We will show by induction on N that this map reduces to an isomorphism modulo p^N for every N, the case $N = 1$ being trivial. Assume now that $N > 1$, that the induced map $\mathrm{W}(S) \otimes_{\mathrm{W}(R)} \mathrm{W}(S)/p^{N-1} \to \mathrm{W}(S \otimes_R S)/p^{N-1}$ is an isomorphism and consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & p^{N-1}\,\mathrm{W}(S) \otimes \mathrm{W}(S)/p^N \arrow[r] \arrow[d] & \mathrm{W}(S) \otimes \mathrm{W}(S)/p^N \arrow[r] \arrow[d] & \mathrm{W}(S) \otimes \mathrm{W}(S)/p^{N-1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & p^{N-1}\,\mathrm{W}(S \otimes_R S)/p^N \arrow[r] & \mathrm{W}(S \otimes_R S)/p^N \arrow[r] & \mathrm{W}(S \otimes_R S)/p^{N-1} \arrow[r] & 0.
\end{tikzcd}$$

Since $\mathrm{W}(S) \otimes \mathrm{W}(S)$ has no p-torsion, this diagram is isomorphic to

$$\begin{tikzcd}
0 \arrow[r] & S \otimes_R S \arrow[r] \arrow[d, "\mathrm{id}"] & \mathrm{W}(S) \otimes \mathrm{W}(S)/p^N \arrow[r] \arrow[d] & \mathrm{W}(S) \otimes \mathrm{W}(S)/p^{N-1} \arrow[r] \arrow[d, "\simeq"] & 0 \\
0 \arrow[r] & S \otimes_R S \arrow[r] & \mathrm{W}(S \otimes_R S)/p^N \arrow[r] & \mathrm{W}(S \otimes_R S)/p^{N-1} \arrow[r] & 0.
\end{tikzcd}$$

On applying the 5-lemma we see that

$$\mathrm{W}_N(S) \otimes_{\mathrm{W}_N(R)} \mathrm{W}_N(S) = \mathrm{W}(S) \otimes_{\mathrm{W}(R)} \mathrm{W}(S)/p^N = \mathrm{W}(S \otimes_R S)/p^N,$$

which finishes the induction step. □

Lemma 8.8. *Let $R \to S$ be a homomorphism of perfect rings. Then for every $N \geq 1$:*

(1) $\mathrm{W}_N(R) \to \mathrm{W}_N(S)$ *is flat if and only if* $R \to S$ *is flat,*
(2) $\mathrm{W}_N(R) \to \mathrm{W}_N(S)$ *is faithful if and only if* $R \to S$ *is faithful.*

(A homomorphism of rings is said to be faithful iff it induces a surjective map on the associated spectra.)

Proof. Let $\mathrm{W}_N(R) \to \mathrm{W}_N(S)$ be flat, and let $M \hookrightarrow N$ be an injection of R-modules. Since every R-module is also a $\mathrm{W}_N(R)$-module via the residue map $\mathrm{W}_N(R) \to R$, we obtain

$$\begin{tikzcd}
M \otimes_{\mathrm{W}_N(R)} \mathrm{W}_N(S) \arrow[r, hook] \arrow[d, "\mathrm{id}"] & N \otimes_{\mathrm{W}_N(R)} \mathrm{W}_N(S) \arrow[d, "\mathrm{id}"] \\
M \otimes_R S \arrow[r, hook, dotted] & N \otimes_R S.
\end{tikzcd}$$

Thus also $R \to S$ is flat. To prove the converse, we use the following theorem of Govorov and Lazard ([Eis95] Theorem A6.6): An R-module is flat if and only if it is the colimit of a *filtered* direct system of free modules. Moreover we note that in this situation the colimit in the category of sets coincides with the colimit in the category of R-modules. So let $(F_i \simeq R^{d_i})_i$ be a filtered direct system having S as its colimit (the d_i may be infinite). I claim that $\mathrm{W}_N(S)$ is the filtered colimit of the induced filtered direct system $(\mathrm{W}_N(F_i) := (\mathrm{W}_N(R)^{d_i})_i$. As noted before, the filtered direct limit of $(\mathrm{W}_N(F_i))_i$ can be calculated in the category of sets, and

there we have $\mathrm{W}_N(F_i) = (R^N)^{d_i}$. But since filtered direct limits commute with finite products we obtain

$$\varinjlim \mathrm{W}_N(F_i) = \varinjlim (R^{d_i})^N = (\varinjlim R^{d_i})^N = S^N = \mathrm{W}_N(S).$$

In other words, the $\mathrm{W}_N(R)$-module $\mathrm{W}_N(S)$ is the colimit of a direct system of free $\mathrm{W}_N(R)$-modules, hence it is flat.

To prove the second statment, we just note that for every ring R the reduction mod p, $\mathrm{W}_N(R) \to R$, induces a bijection between the associated spectra:

$$\operatorname{Spec} R \xrightarrow{\simeq} \operatorname{Spec} \mathrm{W}_N(R).$$

Namely, since p is nilpotent in $\mathrm{W}_N(R)$ it is contained in every prime ideal of $\mathrm{W}_N(R)$. $\square$

Lemma 8.9. *Let $(A_i)_{i\in\mathbb{N}}$ be an inverse system of rings, with all the connecting maps $A_i \to A_{i-1}$ surjective, and let $\hat{A}$ be its limit. Let M be a finitely generated $\hat{A}$-module, write $M_i := M \otimes_{\hat{A}} A_i$ and assume that $M = \varprojlim M_i$. If all the M_i are projective A_i-modules, then M is a projective $\hat{A}$-module.*

Proof. Consider a surjective $\hat{A}$-homomorphism $\pi : \hat{A}^n \twoheadrightarrow M$. We shall show that it splits by constructing a system of compatible splittings of the induced maps $\pi_i : A_i^n \twoheadrightarrow M_i$.

Of course, the maps π_i split, since the M_i are projective by assumption. Our strategy will be to construct *compatible* splittings by induction on i. So assume we have a compatible system of splittings $s_i : M_i \to A_i^n$ up to a certain index i. By tensoring the sequence $0 \to \ker \to A_{i+1} \to A_i \to 0$ with $\pi_{i+1} : (A_{i+1})^n \to M_{i+1}$ we obtain the following diagram of A_{i+1}-modules, with exact rows (L and K being the kernels by definition) and all vertical maps surjective:

$$\begin{tikzcd}
0 \arrow[r] & L \arrow[r] \arrow[d] & A_{i+1}^n \arrow[r] \arrow[d] & A_i^n \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K \arrow[r] & M_{i+1} \arrow[r] & M_i \arrow[r] & 0.
\end{tikzcd}$$

Note that the map $L \to K$ is surjective, for the following reason: Projectivity of M_{i+1} yields a decomposition $A_{i+1}^n = M_{i+1} \oplus C_{i+1}$, C_{i+1} being the kernel of π_{i+1}. By tensoring with A_i we obtain an analogous decomposition $A_i^n = M_i \oplus C_i$, with C_{i+1} surjecting onto $C_i = \ker(\pi_i)$. Now the 5-lemma shows that indeed L surjects onto K.

By induction, for the map $\pi_i : A_i^n \to M_i$ we already have a splitting s_i. By A_{i+1}-projectivity of M_{i+1} we may lift the composition $M_{i+1} \to M_i \to A_i^n$ in order to obtain a map $\widetilde{s_{i+1}} : M_{i+1} \to A_{i+1}$, rendering the right square in the following diagram commutative:

$$\begin{tikzcd}
0 \arrow[r] & L \arrow[r] \arrow[d] & A_{i+1}^n \arrow[r] & A_i^n \arrow[r] & 0 \\
0 \arrow[r] & K \arrow[r] & M_{i+1} \arrow[r] \arrow[u, "\widetilde{s_{i+1}}"] & M_i \arrow[r] \arrow[u, "s_i"] & 0.
\end{tikzcd}$$

In general, $\widetilde{s_{i+1}}$ will not be a splitting of π_{i+1}, but it can be properly adjusted: a diagram chase shows that the difference $\delta_i := (\pi_{i+1} \circ \widetilde{s_{i+1}} - 1)$ is a map $M_{i+1} \to \ker(M_{i+1} \to M_i) = K$. Again by projectivity, we can lift δ_i to $\Delta_i : M_{i+1} \to L \to$

A^n_{i+1} (as remarked above $L \to K$ is surjective). If we set $s_{i+1} := \widetilde{s_{i+1}} - \Delta_i$, we get indeed a splitting of π_{i+1} which still forms a commutative square

$$\begin{tikzcd} A^n_{i+1} \arrow[r] & A^n_i \\ M_{i+1} \arrow[r] \arrow[u, "s_{i+1}"] & M_i. \arrow[u, "s_i"'] \end{tikzcd}$$

Inductively applying this construction, we end up with a projective system of splittings, the limit of which is the desired splitting of π. Thus we are done. □

We are now able to prove that the functor $R \mapsto \mathcal{L}att^n_p(R)$ is a sheaf for the fpqc-topology on the category of perfect rings.

To begin with, note that for any perfect ring R and any $\mathrm{W}(R)$-submodule $M \subset \mathrm{W}(R)[1/p]^n$ satisfying $p^N \mathrm{W}(R)^n \subset M \subset p^{-N} \mathrm{W}(R)^n$ for some N, we have

$$\varprojlim (M \otimes \mathrm{W}(R)/p^i \mathrm{W}(R)) = \varprojlim M/p^i M = \varprojlim M/p^j \mathrm{W}(R)^n = M. \tag{8.1}$$

Here the second equality holds since the respective inverse systems are coinitial, while the third equality follows from the short exact sequence

$$0 \to M/p^j \mathrm{W}(R)^n \to p^{-N} \mathrm{W}(R)^n/p^j \mathrm{W}(R)^n \to p^{-N} \mathrm{W}(R)^n/M \to 0 \quad (j >> 0)$$

upon passage to the inverse limit. Since we already know that $\mathcal{L}att^n_p$ is a Zariski-sheaf, it suffices to consider a faithfully flat homomorphism $R \to S$ of perfect rings, and show that the sequence

$$\mathcal{L}att^n_p(R) \to \mathcal{L}att^n_p(S) \rightrightarrows \mathcal{L}att^n_p(S \otimes_R S) \tag{8.2}$$

is an equalizer.

(1) $\mathcal{L}att^n_p(R) \to \mathcal{L}att^n_p(S)$ is injective: Take $L, L' \in \mathcal{L}att^n_p(R)$ such that $L \otimes_{\mathrm{W}(R)} \mathrm{W}(S) = L' \otimes_{\mathrm{W}(R)} \mathrm{W}(S)$. By Lemma 8.8 we know that $\mathrm{W}_N(R) \to \mathrm{W}_N(S)$ is faithfully flat for every N, which tells us that $L \otimes_{\mathrm{W}(R)} \mathrm{W}_N(R) = L' \otimes_{\mathrm{W}(R)} \mathrm{W}_N(R)$. Using (8.1) this proves $L = L'$.

(2) The difference kernel of $\mathcal{L}att^n_p(S) \rightrightarrows \mathcal{L}att^n_p(S \otimes_R S)$ is equal to $\mathcal{L}att^n_p(R)$: Clearly, $\mathcal{L}att^n_p(R)$ is contained in the difference kernel. Conversely, choose $L \in \mathcal{L}att^n_p(S)$, such that $L' = L \otimes_{\mathrm{W}(S),1} \mathrm{W}(S \otimes_R S)$ equals $L'' = L \otimes_{\mathrm{W}(S),2} \mathrm{W}(S \otimes_R S)$. Note that by the indices 1 and 2, respectively, at the $\otimes$-symbol we indicate which module structure on $\mathrm{W}(S \otimes_R S)$ is under consideration. Then

$$\begin{aligned} &(L \otimes \mathrm{W}_i(S)) \otimes_{\mathrm{W}_i(S),1} (\mathrm{W}_i(S) \otimes_{\mathrm{W}_i(R)} \mathrm{W}_i(S)) = \\ &= (L \otimes \mathrm{W}_i(S)) \otimes_{\mathrm{W}_i(S),2} (\mathrm{W}_i(S) \otimes_{\mathrm{W}_i(R)} \mathrm{W}_i(S)), \end{aligned} \tag{8.3}$$

and similarly

$$\begin{aligned} &(L/p^i(\mathrm{W}(S))^n) \otimes_{\mathrm{W}_{N+i}(S),1} (\mathrm{W}_{N+i}(S) \otimes_{\mathrm{W}_{N+i}(R)} \mathrm{W}_{N+i}(S)) = \\ &= (L/p^i(\mathrm{W}(S))^n) \otimes_{\mathrm{W}_{N+i}(S),2} (\mathrm{W}_{N+i}(S) \otimes_{\mathrm{W}_{N+i}(R)} \mathrm{W}_{N+i}(S)) \end{aligned} \tag{8.4}$$

for i big enough. (here we use Lemma 8.7).

For $i > 2N$ we consider now the diagram of $\mathrm{W}_{i+N}(S)$-modules

$$\begin{tikzcd}
(p^{-N}\,\mathrm{W}(S)^n)/(p^{i+N}\,\mathrm{W}(S)^n) \arrow[r] & p^{-N}\,\mathrm{W}_i(S)^n \arrow[r, equal] & (p^{-N}\,\mathrm{W}(S)^n)/(p^{i-N}\,\mathrm{W}(S)^n) \\
L/p^{N+i}\,\mathrm{W}(S)^n \arrow[r, two heads] \arrow[u, hook] & L\otimes_{\mathrm{W}(S)}\mathrm{W}_i(S) \arrow[r, two heads] \arrow[u] & L/p^{i-N}\,\mathrm{W}(S)^n \arrow[u, hook] \\
(p^{N}\,\mathrm{W}(S)^n)/(p^{i+N}\,\mathrm{W}(S)^n) \arrow[r, equal] \arrow[u, hook] & p^{N}\,\mathrm{W}_i(S)^n \arrow[r] \arrow[u] & (p^{N}\,\mathrm{W}(S)^n)/(p^{i-N}\,\mathrm{W}(S)^n) \arrow[u, hook]
\end{tikzcd}$$

Now (8.3) and (8.4) together with Lemma 8.8 say that this diagram descends to a diagram of $\mathrm{W}_{i+N}(R)$-modules, i.e. we obtain

$$\begin{tikzcd}
(p^{-N}\,\mathrm{W}(R)^n)/(p^{i+N}\,\mathrm{W}(R)^n) \arrow[r] & p^{-N}\,\mathrm{W}_i(R)^n \arrow[r, equal] & (p^{-N}\,\mathrm{W}(R)^n)/(p^{i-N}\,\mathrm{W}(R)^n) \\
P_{N+i} \arrow[r, two heads] \arrow[u, hook] & M_i \arrow[r, two heads] \arrow[u] & P_{i-N} \arrow[u, hook] \\
(p^{N}\,\mathrm{W}(R)^n)/(p^{i+N}\,\mathrm{W}(R)^n) \arrow[r, equal] \arrow[u, hook] & p^{N}\,\mathrm{W}_i(R)^n \arrow[r] \arrow[u] & (p^{N}\,\mathrm{W}(R)^n)/(p^{i-N}\,\mathrm{W}(R)^n) \arrow[u, hook]
\end{tikzcd}$$

We thus have two cofinal systems of $\mathrm{W}(R)$-modules, (M_i) and (P_i), whose inverse limit is a $\mathrm{W}(R)$-module M. I claim that this is the desired $\mathrm{W}(R)$-lattice. First observe that for N big enough we have an exact sequence

$$0 \to p^N\,\mathrm{W}(R)^n \hookrightarrow M \to P_N \to 0,$$

as we see by taking the inverse limit over $i > N$ of the sequence

$$0 \to p^N\,\mathrm{W}(R)^n/p^i\,\mathrm{W}(R)^n \hookrightarrow P_i \to P_i/p^N\,\mathrm{W}(R)^n = P_N \to 0.$$

Since P_i is finitely generated (by faithfully flat descent) as well as $p^N\,\mathrm{W}(R)^n$, also M is finitely generated. On the other hand, since

$$0 \to p^N L \otimes_{\mathrm{W}(S)} \mathrm{W}_i(S) \to L \otimes_{\mathrm{W}(S)} \mathrm{W}_{i+N}(S) \to L \otimes_{\mathrm{W}(S)} \mathrm{W}_N(S) \to 0$$

is exact, we obtain by faithfully flat descent a short exact sequence

$$0 \to p^N M_i \to M_{i+N} \to M_N \to 0.$$

Passing to the inverse limit over i we obtain

$$0 \to p^N M \to M \to M_N \to 0,$$

and thus $M \otimes_{\mathrm{W}(R)} \mathrm{W}_N(R) = M_N$, which is a projective $\mathrm{W}_N(R)$-module, by faithfully flat descent. Hence we have arrived at a situation where Lemma 8.9 applies, proving that $M = \varprojlim(M \otimes_{\mathrm{W}(R)} \mathrm{W}_N(R))$ is a $\mathrm{W}(R)$-lattice. Clearly, $(M \otimes_{\mathrm{W}(R)} \mathrm{W}(S)) \otimes_{\mathrm{W}(S)} \mathrm{W}_N(S) = M_N \otimes \mathrm{W}_N(S) = L \otimes_{\mathrm{W}(S)} \mathrm{W}_N(S)$. Taking the limit over N we obtain $M \otimes_{\mathrm{W}(R)} \mathrm{W}(S) = L$, which finishes the proof. $\square$

From this theorem we obtain two corollaries.

Corollary 8.10. *Let R be a perfect k-algebra and let $L \subset \mathrm{W}(R)[1/p]^n$ be a $\mathrm{W}(R)$-submodule. Then the following are equivalent:*

1. *The submodule L is a lattice.*
2. *Zariski-locally on R, L is a free $\mathrm{W}(R)$-submodule of rank n (i.e. there exist $f_1, \ldots, f_r \in R$ such that $(f_1, \ldots, f_r) = \mathrm{W}(R)$ and for all i, $L \otimes_{\mathrm{W}(R)} \mathrm{W}(R_{f_i})$ is free of rank n and $L \otimes_{\mathrm{W}(R)} \mathrm{W}(R)[1/p] = \mathrm{W}(R)[1/p]^n$.*

(3) *fpqc-locally on R, L is a free* $\mathrm{W}(R)$*-submodule of rank n (i.e. there exists a faithfully flat ring homomorphisms $R \to S$ such that $L \otimes_{\mathrm{W}(R)} \mathrm{W}(S)$ is free of rank n and $L \otimes_{\mathrm{W}(R)} \mathrm{W}(R)[1/p] = \mathrm{W}(R)[1/p]^n$.*

Proof. This follows immediately from Theorem 8.6. □

It is not clear to me whether there is a good translation of condition (4) of Theorem 8.2 to the Witt vector setting. The obvious obstacle is the fact that $\mathrm{W}(R)$ does not carry a structure of R-module.

Corollary 8.11. *The fpqc-sheaf $\mathcal{L}att_p^{n,0}$ is equal to the restriction of the p-adic affine Grassmannian $\mathcal{G}rass_p$ to the category of perfect k-algebras.*

Proof. The presheaf $R \mapsto \mathrm{Sl}_n(\mathrm{W}(R)[1/p])/\,\mathrm{Sl}_n(\mathrm{W}(R))$ coincides with the presheaf $R \mapsto$ { free special lattices of rank n over $\mathrm{W}(R)$ } on the category of perfect k-algebras. Thus it suffices to prove that for any presheaf F on the fpqc-site over k the processes of 'sheafification' and 'restriction to the category of perfect k-algebras' commute. Let R be a perfect k-algebra and let $\{U_i \to \operatorname{Spec} R\}$ be a covering (on the fpqc-site over k). Refining the covering we may assume that the U_i are all affine. For every i denote by U_i^{perf} the perfection of U_i. Then the morphisms $U_i^{perf} \to \operatorname{Spec} R$ are still flat and jointly surjective and thus define a refinement of $\{U_i \to \operatorname{Spec} R\}$, which is by definition also a covering in the fpqc-site on the category of perfect k-algebras. Now the claim follows from Lemma 9.1 in the appendix. □

9. Appendix: fpqc-sheaves

In this appendix we collect some general results on fpqc-sheaves which are used throughout the preceding chapters. In particular, questions concerning the existence of sheafifications on the fpqc-site are often ignored, but I have tried to give precise arguments why the desired objects really exist in our setting.

9.1. **fpqc-sheaves and sheafifications.** Let $\mathcal{C}$ be the category of schemes. By a presheaf on $\mathcal{C}$ we mean simply a functor on the category of schemes.

Lemma 9.1. *Let $\mathcal{D} \subset \mathcal{C}$ be an inclusion of sites, such that fiber products in $\mathcal{D}$ are mapped to fiber products in $\mathcal{C}$. Assume that for every covering $\mathcal{U} = \{U_i \to X\}$ in $\mathcal{C}$ of an object $X \in \mathcal{D}$ there exists a refinement $\mathcal{V} = \{V_i \to X\}$ of $\mathcal{U}$ with $V_i \in \mathcal{D}$ such that $\mathcal{V}$ is also a covering of X in $\mathcal{D}$.*

Claim: if F has a sheafifcation F^a, then $F^a|_{\mathcal{D}}$ is a (the) sheafification of $F|_{\mathcal{D}}$.

Proof. Let F^a be the sheafification of F. Clearly, $F^a|_{\mathcal{D}}$ is a sheaf on $\mathcal{D}$, whence the canonical map $(F|_{\mathcal{D}})^a \to (F^a)|_{\mathcal{D}}$. To prove that this is an isomorphism, we check that the morphism $F|_{\mathcal{D}} \to (F^a)|_{\mathcal{D}}$ is a sheafification on $\mathcal{D}$. Thus let $X \in \mathcal{D}$ and let $\xi, \eta \in F(X)$ such that their images in $F^a(X)$ coincide. By definition of sheafification there exists a covering (in $\mathcal{C}$) of X on which ξ and η coincide. But by assumption this covering can be refined so to obtain a covering of X in $\mathcal{D}$ on which ξ and η coincide a fortiori. On the other hand, every element $\xi \in F^a(X)$ can be represented locally (on a covering in $\mathcal{C}$) by sections of F. Refining this covering, we see that ξ can be represented on a covering in $\mathcal{D}$ by sections of F. □

Theorem 9.2 (Vistoli [Vis08]). *Let F be a presheaf on $\mathcal{C}$. Assume that F is a sheaf for the Zariski topology. Then F is an fpqc-sheaf on $\mathcal{C}$ if and only if for every faithfully flat homomorphism of affine schemes $Y \to X$ the sequence*

$$F(X) \to F(Y) \rightrightarrows F(Y \times_X Y) \tag{9.1}$$

is an equalizer. □

Proposition 9.3. *Let F be a presheaf on $\mathcal{C}$. Assume that F satisfies the following two conditions:*

(1) *for every faithfully flat morphism of* affine *schemes $Y \to X$ the sequence*
$$F(X) \to F(Y) \rightrightarrows F(Y \times_X Y)$$
is an equalizer, and
(2) *for every finite collection of affine schemes $Y_1, \dots, Y_n$ we have*
$$F(Y_1 \coprod \cdots \coprod Y_n) = F(Y_1) \times \cdots \times F(Y_n).$$

Then the Zariski-sheafification F^a of F is an fpqc-sheaf. In particular, F^a is an fpqc-sheafification of F. Moreover, the natural transformation $F \to F^a$ restricts to an isomorphism on the category of affine schemes.

Proof. In view of Theorem 9.2 we only have to prove that the condition in (1) of the present proposition remains valid after Zariski-sheafification. Thus it will suffice to prove the last assertion, namely that the natural map $F(X) \to F^a(X)$ is indeed an isomorphism for every affine X. To this end, for an arbitrary scheme X and any Zariski-covering $\mathcal{U}$ of X let $K(\mathcal{U})$ be the difference kernel of $F(\mathcal{U}) \rightrightarrows F(\mathcal{U} \times_X \mathcal{U})$. If we set $F'(X) = \varinjlim_{\mathcal{U}} K(\mathcal{U})$, where the colimit is taken over all Zariski-coverings of X, then F' will be a separated presheaf. Applying this procedure twice, i.e. forming F'', will yield a *sheaf*, and indeed F'' is equal to the Zariski-sheafification F^a of F. Now observe the following: if X is affine, there is a cofinal subsystem of all Zariski coverings of X given by those coverings which consist of only *finitely many affines*. Thus, using assumption (2),
$$F'(X) = \varinjlim_{Y \to X} \ker(F(Y) \rightrightarrows F(Y \times_X Y)),$$
where now the limit is taken over a certain family of faithfully flat morphisms $Y \to X$ of affine schemes. But by assumption (1) for every such $Y \to X$ we have $F(X) = \ker(F(Y) \rightrightarrows F(Y \times_X Y))$, whence $F'(X) = F(X)$. This implies $F^a(X) = F(X)$, as desired. □

Corollary 9.4. *Let F be as in the proposition. Then the restriction of F to the site of affine schemes (with arbitrary covering families consisting of affine schemes) is a sheaf for the fpqc-topology.* □

The preceding discussion shows that the category of k-spaces in the sense of section 2 is equivalent to the category of functors on affine k-schemes which satisfy the conditions (1) and (2) of Proposition 9.3. Inverse equivalences are given by restriction resp. by passing to the associated Zariski-sheaf. In their paper [BL94], Beauville and Laszlo indeed define a k-space to be a functor on the category of affine k-schemes which satisfies condition (1). On the other hand, they do not require condition (2), which, however, does not seem to be automatic.

The following proposition shows that indeed every directed system of k-schemes gives rise to an ind-scheme (i.e. the colimit in the category of k-spaces exists).

Proposition 9.5. *A functor which is represented by a directed system of schemes admits an fpqc-sheafification. Indeed, it suffices to take its Zariski-sheafification, which is then automatically an fpqc-sheaf(ification). Moreover, the restriction of this sheafification to the category of affine schemes coincides with the original presheaf defined by the inductive system of schemes.*

Proof. We have to check that such a functor satisfies the assumptions (1) and (2) of Proposition 9.3.

To this end, let (X_i) be a direct system of schemes and let $\varinjlim X_i$ be its colimit in the category of presheaves. Let $T_1, \ldots, T_n$ be affine schemes. Then we have

$$\begin{aligned}(\varinjlim X_i)(T_1 \coprod \cdots \coprod T_n) &= \varinjlim (X_i(T_1 \coprod \cdots \coprod T_n)) = \\ &= \varinjlim (X_i(T_1) \times \cdots \times X_i(T_n)) = (\varinjlim X_i)(T_1) \times \cdots \times (\varinjlim X_i)(T_n),\end{aligned}$$

which is condition (2). It remains to check exactness of the sequence

$$(\varinjlim X_i)(R) \to (\varinjlim X_i)(S) \rightrightarrows (\varinjlim X_i)(S \otimes_R S),$$

where $R \to S$ is a faithfully flat homomorphism of rings. Thus let $P \in (\varinjlim X_i)(S)$ such that both images of P in $(\varinjlim X_i)(S \otimes_R S)$ coincide. Assume that P is represented by an element $P' \in X_i(S)$. By definition of the inductive limit, there exists some $i \leq j \in I$ such that that the induced objects in $X_j(S \otimes_R S)$ coincide. Now we can use the exactness of the sequence

$$X_j(R) \to X_j(S) \rightrightarrows X_j(S \otimes_R S)$$

to obtain an R-valued point of X_j, and hence an R-valued point of $\varinjlim X_i$ which induces P. This shows that the difference kernel of the right hand maps is precisely the image of the left hand map. Injectivity of the left hand map is proved likewise, which shows that condition (1) holds as well. $\square$

Proposition 9.5 says that if we restrict the functor direct-limit $\varinjlim X_i$ to the category of affine schemes (or more generally: quasi-compact schemes), then it is already a sheaf for the fpqc-topology. This is Beauville and Laszlo's point of view.

9.2. **fpqc-sheafifications in general.** Contrary to what Vistoli claims in [Vis08] Theorem 2.64, arbitrary functors on the category of k-schemes do not in general admit an fpqc-sheafification. An example of such a functor is described by Waterhouse in [Wat75]. As Waterhouse explains, the general problem with constructing an fpqc-sheafification of an arbitrary functor is that one is forced to consider direct limits over 'all' fpqc-coverings of a given scheme. However, the entirety of 'all' fpqc-coverings will not be a set, but a proper class. One way out of this problem would be to restrict to a fixed universe, which will have the drawback that sheafifications depend on the particular choice of the universe. On the other hand, Waterhouse proves that for 'basically bounded' functors it suffices to look at direct limits over certain *sets* of fpqc-coverings, which resolves the above described set-theoretical problems. The purpose of this section is to check that the quotient-functor $\mathrm{L}_p \mathrm{Sl}_n / \mathrm{L}_p^+ \mathrm{Sl}_n$ is basically bounded, and thus has a well-defined fpqc-sheafification.

Let m be a cardinal number not less than the cardinality of k, fix a set S of cardinality m, and let (k-Alg(m)) be the category of k-algebras whose underlying set is contained in S. Let (k-Alg) denote the category of 'all' k-algebras, and let $j : (k\text{-Alg(m)}) \hookrightarrow (k\text{-Alg})$ be the inclusion. For any set-valued functor on the category of k-algebras, let j^* denote the restriction to (k-Alg(m)). Right-adjoint to j^* is the Kan extension j_* along $(k\text{-Alg(m)}) \hookrightarrow (k\text{-Alg})$.

Definition 9.6. A functor F on the category of k-algebras is *m-based* if it has the form j_*G for some functor G on (k-Alg(m)). A functor is *basically bounded* if there exists a cardinal m such that it is m-based.

Theorem 9.7 ([Wat75], Corollary 5.2)**.** *If a functor F on the category of k-algebras is m-based, then it has an fpqc-sheafification. More precisely, if $j^*F \to G$ is a sheafification for the fpqc-topology on (k-Alg(m)), then $F = j_*j^*F \to j_*G$ is an fpqc-sheafification on (k-Alg).* □

We use the following two observations by Waterhouse: (1) A functor which is represented by an affine scheme whose underlying ring has cardinality $\leq m$ is m-based. (2) The Kan extension j_* preserves colimits, and in particular, the colimit over a system of basically bounded functors is again basically bounded.

Theorem 9.8. *The functor-quotient $\mathrm{L}_p \mathrm{Sl}_n \,/\, \mathrm{L}_p^+ \mathrm{Sl}_n$ is basically bounded, and hence has a well-defined fpqc-sheafification. Thus the p-adic affine Grassmannian in our sense exists.*

Proof. By (2) above, $\mathrm{L}_p \mathrm{Sl}_n$ as well as $\mathrm{L}_p^+ \mathrm{Sl}_n$ are basically bounded functors on the category of k-algebras. Since $\mathrm{L}_p \mathrm{Sl}_n \,/\, \mathrm{L}_p^+ \mathrm{Sl}_n$ is the colimit of a system

$$\mathrm{L}_p \mathrm{Sl}_n \times \mathrm{L}_p^+ \mathrm{Sl}_n \rightrightarrows \mathrm{L}_p \mathrm{Sl}_n,$$

it is basically bounded, too. By Waterhouse's theorem, it thus has an fpqc-sheafification $\mathcal{G}rass'_p$ on the category of k-algebras. Moreover, since the functor-quotient $\mathrm{L}_p \mathrm{Sl}_n \,/\, \mathrm{L}_p^+ \mathrm{Sl}_n$ satisfies condition (2) of Proposition 9.3, so does $\mathcal{G}rass'_p$: namely, the set of fpqc-covers inside (k-Alg(m)) of $\coprod T_i$ (finite disjoint union) is in natural bijection with the product $\prod\{$ fpqc-covers of T_i $\}$, and direct limits (used to compute sheafifications) commute with finite products. All in all, $\mathcal{G}rass'_p$ satisfies the hypotheses of Proposition 9.3, and its Zariski-sheafification $\mathcal{G}rass_p$ will be the desired fpqc-sheafification of $\mathrm{L}_p \mathrm{Sl}_n \,/\, \mathrm{L}_p^+ \mathrm{Sl}_n$ on the category of k-schemes. □

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